These are brief notes for the lecture on Monday September 13, 2010: they are not complete, but they are a guide to what I want to say today. They are not guaranteed to be correct.

2.2. Matrix Inverses, continued

Recall the matrices corresponding to elementary row operations Denoting rows r and s by R_r and R_s, the row operations are:

(1) Interchange rows R_r and R_s of a matrix.

(2) For a non-zero $c \in \mathbb{R}$, replace R_r by cR_r.

(3) Replace R_r (row r) by $R_r - cR_s$

Now, if A has column form $[\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n]$ and B has the appropriate size, then BA has column form $[B\underline{a}_1, B\underline{a}_2, \ldots, B\underline{a}_n]$.

Hence, if for each of the elementary row operations we can find a matrix which represents that operation on a vector, it will also represent the same operation on A.

(1) The matrix M having (i, j)-entry m_{ij} representing interchanging R_r and R_s agrees with the identity matrix except for four entries:

$$\begin{aligned} m_{rr} &= 0 & \qquad m_{rs} &= 1 \\ m_{sr} &= 1 & \qquad m_{ss} &= 0 \end{aligned}$$

(2) The matrix M representing replacing row R_r by cR_r agrees with the identity matrix except for the (r, r)-entry, which is $m_{rr} = c$.

(3) The matrix M representing replacing R_r by $R_r - cR_s$ agrees with the identity matrix except for the the (r, s)-entry, which is $m_{rs} = -c$.

Exercise: for a matrix having 4 rows, write down the elementary matrices which

(1) switches R_1 and R_3

(2) replace R_2 by $3R_2$

(3) replaces R_2 by $R_2 + 7R_4$

Exercise: write down the inverse for each of the elementary matrices above.

Note: Suppose that we have a matrix A and that we perform a sequence of elementary row operations, having matrices E_1, E_2, $\ldots$,E_k. Then applying just the first operation we obtain a matrix E_1A: applying the second operation, you obtain E_2E_1A, applying the third, $E_3E_2E_1A$, and so after applying all of the operations we end up with

$$B = E_kE_{k-1}\ldots E_2E_1A.$$

We've seen already that an $n \times n$ matrix A is invertible if and only if every equation $A\underline{x} = \underline{b}$ has a unique solution: this is true if and only if the row reduced echelon form of A has a pivot in every row and column, which is if and only if the row reduced echelon form is I_n.

Hence

$$I_n = E_kE_{k-1}\ldots E_2E_1A$$

and we see that

$$\begin{aligned} A^{-1} &= E_kE_{k-1}\ldots E_2E_1 \\ &\quad E_kE_{k-1}\ldots E_2E_1I_n. \end{aligned}$$

This gives us a way to compute A^{-1} by hand: augment A by the *matrix* I_n, and row reduce the resulting $n \times 2n$ matrix. If A is invertible, then of course, the first half of the columns will row reduce to the identity. The remaining columns will give A^{-1}.

We state this as a theorem:

THEOREM 7. *An $n \times n$ matrix A is invertible if and only if $A \sim I_n$, in which case the sequence of elementary row operations which transform A to the identity also transform the identity matrix I_n to A^{-1}.*

Example: Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 3 & 5 & 1 \end{pmatrix}$. Find A^{-1}.

2.3. Characterization of Invertible Matrices

At the beginning of our discussion of inverses, we discussed inverses in terms of linear transformations. We formalize that discussion now.

THEOREM 8. *Let A be an $n \times n$ matrix (and this is essential): the following conditions are equivalent:*

(1) *A is invertible.*

(2) $A \sim I_n$.

(3) *A has n pivots.*

(4) *$A\underline{x} = \underline{0}$ has only the trivial solution.*

(5) *The columns of A are linearly independent.*

(6) *The linear transformation $T : \underline{x} \longmapsto A\underline{x}$ is 1-1.*

(7) *For every $\underline{b} \in \mathbb{R}^n$, the equation $A\underline{x} = \underline{b}$ has at least one solution.*

(8) *The columns of A span $\mathbb{R}^n$*

(9) *The linear transformation $T : \underline{x} \longmapsto A\underline{x}$ is onto.*

(10) *$\exists C$ so that $CA = I_n$.*

(11) *$\exists D$ so that $DA = I_n$.*

(12) *A^T is invertible.*

Proof: To prove a number of statements are equivalent, it is often easiest to show a chain of beginning and ending at one of the statements. Here, it is easiest to do the following chains:

$$(1) \implies (10) \implies (4) \implies (3) \implies (2) \implies (1)$$
$$(1) \implies (11) \implies (7) \implies (1)$$

$$
\begin{gathered}
(7) \iff (8) \iff (9) \\
(4) \iff (5) \iff (6) \\
(1) \iff (12)
\end{gathered}
$$

Finally to bring things full circle, we summarize how we started the discussion of inverses:

THEOREM 9. *Let* $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be a linear transformation, and let* A *be the corresponding matrix. Then* T *is invertible if and only if* A *is invertible, in which case* $T^{-1}\underline{x} = A^{-1}\underline{x}$.