These are brief notes for the lecture on Friday September 17, 2010: they are not complete, but they are a guide to what I want to say today. They are not guaranteed to be correct.

2.4. Subspaces of $\mathbb{R}^n$

In Chapter 4 we will meet general vector spaces, and study subspaces of them: in order to make the ideas we meet there more concrete, we'll briefly discuss subspaces and dimension of subspaces here.

A *subspace* of $\mathbb{R}^n$ is a set of vectors in $\mathbb{R}^n$ which looks like a copy of $\mathbb{R}^m$ in its own right: for example, if we are in $\mathbb{R}^3$, any plane through the origin looks a lot like $\mathbb{R}^2$: it is "2-dimensional", the sum of any two vectors in the plane is in the plane, and any scalar multiple of a vector in the plane is in the plane. Similarly, any line through the origin looks a lot like $\mathbb{R} = \mathbb{R}^1$. How can we formalize this notion, and get our hands on what "dimension" means?

DEFINITION. *A* subspace *of* $\mathbb{R}^n$ *is a set* H *of vectors in* $\mathbb{R}^n$ *with the following three properties:*

(1) *The zero vector* $\underline{0}$ *from* $\mathbb{R}^n$ *is in* H*.*

(2) *For every* $\underline{u}, \underline{v} \in H$*,* $\underline{u} + \underline{v} \in H$*.*

(3) *For each* $\underline{u} \in H$ *and* $c \in \mathbb{R}$*,* $c\underline{u} \in H$*.*

Example 1: If $\underline{v}_1$ and $\underline{v}_2$ are in $\mathbb{R}^n$, then $\text{Span}(\underline{v}_1, \underline{v}_2)$ is a subspace of $\mathbb{R}^n$. To check this, note first that $\underline{0} = 0\underline{v}_1 + 0\underline{v}_2$.
Now check it is closed under addition:

Now check it is closed under scalar multiplication:

Example 2: A line *not* through the origin is not a subspace: it doesn't contain $\underline{0}$. Also, it is not closed under scalar multiplication or addition.

Example 3: For $\underline{v}_1, \underline{v}_2, \dots \underline{v}_p \in \mathbb{R}^n$, Span$(\underline{v}_1, \underline{v}_2, \dots \underline{v}_p)$ is a subspace of $\mathbb{R}^n$. We will refer to this as *the subspace spanned* (or *generated*) by $\underline{v}_1, \underline{v}_2, \dots \underline{v}_p$.

An important special case of this example is the following:

DEFINITION. *The* column space *of a matrix A is the set Col(A) of all linear combinations of the columns of A.*

That is, Col(A) is the span of the columns of A.

The other common way in which subspaces arise is as the solution set to a homogeneous system of equations:

DEFINITION. *The* null space *of a matrix A is the set Nul(A) of all solutions to the homogeneous equation $A\underline{x} = \underline{0}$.*

THEOREM 12. *The null space of a $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.*

2.5. Basis

The examples we have seen of subspaces of $\mathbb{R}^n$ have the property that they can be expressed as the span of a set of vectors. (To see that this is true of the null space, row reduce the matrix, and express the null space in parametric form).

Now, if the list of vectors which we are using to express the subspace is not linearly independent, then it is redundant, in the sense that we don't need all of the vectors to span everything.

If a set of vectors spans a subspace *and* is linearly independent, then we call it a *basis*.

Fact: every basis for a given subspace has the same number of vectors in it.

DEFINITION. *Let V be a subspace of $\mathbb{R}^n$. The* dimension *of V is the number of vectors in a basis for V.*

2.6. Determinants

Determine when the following matrices are invertible, using row reduction:
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$.

Definition. *If A is the 3×3 matrix above, then* $\det(A) = aei + bfg + cdh - afh - bdi - ceg$.

Note: this can be viewed as the sum of the products of forward diagonals minus the sum of the products of backward diagonals. BUT NOTE ALSO: THIS PATTERN DISAPPEARS IF $n \geq 4$.
Fact: A 3×3 A is invertible if and only if $\det(A) \neq 0$.

There are various equivalent definitions of the determinant of a square matrix. The one given in the text is just one of many. For computational purposes, we shall use the following:

DEFINITION. *Suppose that A is a square matrix. Reduce A to echelon form (using only row-replacement and row-switching). Then the* determinant *of A, denoted by* $\det(A)$ *or (sometimes $\Delta(A)$) is $(-1)^r$ times the product of the diagonal elements of the echelon form, where r is the number of row-switches performed.*

We immediately see that A is invertible if and only if its echelon form has no zeros on the diagonal: hence we have the following condition:

COROLLARY. *A is invertible if and only if* $\det(A) \neq 0$.

COROLLARY. *If A is triangular, then* $\det(A) = a_{11}a_{22}\ldots a_{nn}$.

COROLLARY. *If A has two rows the same, then* $\det(A) = 0$.

Indeed, if A has two rows the same, then row reduction will first perform the same actions on both of them (keeping them equal) and then will subtract one of them from the other, leaving a row of zeros.

Since A is invertible if and only if A^T is invertible, and since A is invertible if and only if $\det(A) \neq 0$, we see that $\det(A^T) = 0 \iff \det(A) = 0$. Hence

COROLLARY. *If A has two columns the same, then* $\det(A) = 0$.

Fact: $\det(A^T) = \det(A)$.

THEOREM 13. *If A and B are $n \times n$ matrices, then* $\det(A) = \det(B)$.