These are brief notes for the lecture on Monday September 20, 2010: they are not complete, but they are a guide to what I want to say today. They are not guaranteed to be correct.

### 3.1. Determinants

Determine when the following matrices are invertible, using row reduction:
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$.

Definition. *If $A$ is the $3\times 3$ matrix above, then* $\det(A) = aei + bfg + cdh - afh - bdi - ceg$.

Note: this can be viewed as the sum of the products of forward diagonals minus the sum of the products of backward diagonals. BUT NOTE ALSO: THIS PATTERN DISAPPEARS IF $n \geq 4$.
**Fact**: A $3 \times 3$ $A$ is invertible if and only if $\det(A) \neq 0$.

There are various equivalent definitions of the determinant of a square matrix. The one given in the text is just one of many. For computational purposes, we shall use the following:

DEFINITION. *Suppose that $A$ is a square matrix. Reduce $A$ to echelon form (using only row-replacement and row-switching). Then the* determinant *of $A$, denoted by* $\det(A)$ *or (sometimes $\Delta(A)$) is $(-1)^r$ times the product of the diagonal elements of the echelon form, where $r$ is the number of row-switches performed.*

We immediately see that $A$ is invertible if and only if its echelon form has no zeros on the diagonal: hence we have the following condition:

COROLLARY. *$A$ is invertible if and only if* $\det(A) \neq 0$.

COROLLARY. *If $A$ is triangular, then* $\det(A) = a_{11}a_{22}\ldots a_{nn}$.

COROLLARY. *If $A$ has two rows the same, then* $\det(A) = 0$.

Indeed, if $A$ has two rows the same, then row reduction will first perform the same actions on both of them (keeping them equal) and then will subtract one of them from the other, leaving a row of zeros.

Since $A$ is invertible if and only if $A^T$ is invertible, and since $A$ is invertible if and only if $\det(A) \neq 0$, we see that $\det(A^T) = 0 \iff \det(A) = 0$. Hence

COROLLARY. *If $A$ has two columns the same, then* $\det(A) = 0$.

**Fact**: $\det(A^T) = \det(A)$.

THEOREM 1. *If $A$ and $B$ are $n \times n$ matrices, then* $\det(AB) = \det(A)\det(B)$.