These are brief notes for the lecture on Wednesday September 29, 2010: they are not complete, but they are a guide to what I want to say today. They are not guaranteed to be correct.

4.1. Vector Spaces and Subspaces, continued

We'll now see several examples of sets which are familiar from other areas of mathematics, but which can be viewed as vector spaces.

Example Let $n \geq 0$ be an integer. The set $\mathbb{P}_n$ of polynomials of degree at most n consists of all polynomials of the form

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n$$

where the coefficients $a_0, a_1, \ldots a_n$ and the variable t are real numbers. The *degree* of $p(t)$ is the highest power of t whose coefficient is not zero. If $p(t) = a_0 \neq 0$, then the degree of $p(t)$ is zero. If all the coefficients of $p(t)$ are zero, then we call $p(t)$ the *zero polynomial*: its degree is technically speaking undefined, but we include it in the set $\mathbb{P}_n$ too.

We can add two polynomials:

We can multiply a polynomial by a scalar:

The set $\mathbb{P}_n$ is a vector space. The zero polynomial is the zero vector.

Example: Let $\mathbb{P}_n$ be the set of all polynomials, that is $\mathbb{P} = \bigcup_{n \geq 0} \mathbb{P}_n$. Then $\mathbb{P}$ is also a vector space. Note also that $\mathbb{P}_0 \subseteq \mathbb{P}_1 \subseteq \mathbb{P}_2 \ldots$ and for each $n \geq 0$, $\mathbb{P}_n \subseteq \mathbb{P}$.

Example: The subset of $\mathbb{P}_n$ consisting of those polynomials which satisfy $p(1) = 0$ and $p'(\pi) = 0$. It is clear that if we add two polynomials p and q which are both zero at $t = 1$, then the same is true of their sum. Likewise if their derivatives are zero at $t = \pi$ then the same is true of their sum. Checking the rest of the conditions is left as an exercise.

Example Let $C(0, 1)$ be the set of continuous functions defined on the interval $(0, 1)$. Then we can add two continuous functions and get a continuous function: we can multiply a continuous function by a constant and we still get a continuous function: and addition and multiplication "work nicely": so $C(0, 1)$ is also a vector space. Here the zero vector is the function which is zero on the interval $(0, 1)$.

Example Let V be the set of functions $f(t) \in C(0,1)$ with the property that

$$\int_0^1 f(t)\ \mathrm{d}t = 0.$$

Then the sum of two functions with integral zero is a function whose integral is zero. If we multiply f by a scalar, we still get a function whose integral is zero. Addition and multiplication "work nicely", so this is also a vector space. What is the zero vector?

Example Let V be the set of functions $f \in C(0,1)$ for which $f(1/2) = 0$. Is this a vector space?

Example: Let V be the set of functions $f \in C(0,1)$ for which $f(1/2) = 1$. Is this a vector space?

Example: Let V be the set of polynomials of degree exactly n. Is this a vector space?

Example: Let $\mathbb{M}_{m,n}$ denote the set of $m \times n$ matrices. Does this form a vector space?

Subspaces of a vector space

Definition. *If V is a subspace with respect to $+, \cdot$, with zero vector $\underline{0}$, then a set $H \subseteq V$ is a* subspace *of V if*

(1) $\underline{0} \in H$

(2) *For every $\underline{u}, \underline{v} \in H$, $\underline{u} + \underline{v} \in H$.*

(3) *For every $\underline{u} \in H$ and $c \in \mathbb{R}$, $c\underline{u} \in H$.*

Example: For any vector space V with zero vector $\underline{0}$, the set $\{\underline{0}\}$ is a subspace of V.

Example: If $m < n$ the $\mathbb{P}_m$ is a subspace of $\mathbb{P}_n$.

Note: $\mathbb{R}^2$ is *not* a subspace of $\mathbb{R}^3$. Indeed, $\mathbb{R}^2$ is not even a *subset* of $\mathbb{R}^3$. However, a plane through the origin in $\mathbb{R}^3$ *is* a subspace of $\mathbb{R}^3$.

Recall the definitions of linear combinations and span:

Definition. *Suppose that $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k \in V$ and $c_1, c_2, \ldots c_k \in \mathbb{R}$. Then*

$$\sum_{i=1}^{k} c_i \underline{v}_i$$

is the linear combination of $\underline{v}_1, \ldots, \underline{v}_k$ with weights $c_1, \ldots c_k$.

Definition. *Span$(\underline{v}_1, \ldots, \underline{v}_k)$ denotes the set of all linear combinations of $\underline{v}_1, \ldots, \underline{v}_k$.*

Theorem 1. *If V is a vector space and if $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k \in V$, then $H =$ Span$(\underline{v}_1, \ldots, \underline{v}_k)$ is a subspace of V.*

Proof:

4.2. Null Spaces, Column Spaces and Linear Transformations

Recall the definition of the null space of a matrix:

DEFINITION. *Let A be a $m \times n$ matrix, so that the transformation $\underline{x} \mapsto A\underline{x}$ maps $\mathbb{R}^n$ to $\mathbb{R}^m$. The* null space *of A is defined to be*

$$Nul\ A = \{\underline{x} : \underline{x} \in \mathbb{R}^n \text{ and } A\underline{x} = \underline{0}\}.$$

That is, it is those elements of $\mathbb{R}^n$ which are mapped to $\underline{0}$ by A.

Note that Nul A is a subset of $\mathbb{R}^n$.

THEOREM 2. *If A is an $m \times n$ matrix, then Nul A is a subspace of $\mathbb{R}^n$.*

Proof:

Example: Let $H = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} : a - 2b + 5c = d \text{ and } c - a = b \right\}$. Show that H is a subspace of $\mathbb{R}^4$ by expressing this as a null space of a matrix. Find a spanning set for this H.

Example: Find a spanning set for the null space of

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & -3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$$