These are brief notes for the lecture on Friday October 8, 2010: they are not complete, but they are a guide to what I want to say today. They are not guaranteed to be correct.

4.3. Linear Independence in Vector Spaces; Bases

DEFINITION. *Suppose that V is a vector space, and that $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_p \in V$. This sequence of vectors is a* basis *for V if it is linearly independent and spans V. A basis for a subspace $H < V$ is a sequence of vectors in H which is linearly independent and spans H.*

Note: often a *set* of vectors is described as a basis, but then the discussion uses that $\underline{b}_1$ is the first vector in the set, $\underline{b}_2$ is the second vector in the set, etc. This makes it slightly better to refer to a sequence of vectors as being a basis (or as the book puts it, an "indexed set", which is non-standard terminology). For our purposes, we will allow a basis to be either a sequence or a set as is most convenient.

Example: Let $A = [\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n]$ be an invertible $n \times n$ matrix. What does the Invertible Matrix Theorem say about $\{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n\}$?

Example: $\{\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n\}$ is a basis for $\mathbb{R}^n$. Why?

Example: $\{1, x, x^2, \ldots, x^n\}$ is a basis for $\mathbb{P}_n$. Why?

THEOREM 5. *Let V be a vector space, and $S = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_p\} \subseteq V$, and let $H = Span(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_p)$.*

(1) *If there exists k so that $\underline{v}_k$ is a linear combination of the other vectors in S, then*

$$H = Span(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_{k-1}, \underline{v}_{k+1}, \ldots, \underline{v}_p)$$

(2) *If $H \neq \{\underline{0}\}$ then some subset of S is a basis for H.*

Proof:

(1) Suppose that $\underline{u} \in H$. We need to show that $\underline{u}$ is a linear combination of vectors in $\underline{v}_1, \ldots, \underline{v}_{k-1}, \underline{v}_{k+1} \ldots, \underline{v}_n$. We know that it is a linear combination of vectors in S.

(2) If S is linearly independent, then it is a basis. Otherwise, there is a non-trivial linear combination of vectors in S giving $\underline{0}$, and hence there is some vector in S which can be written as a linear combination of the others. Hence we can replace S by a smaller set S' which still spans H. Clearly we can continue this process, and it has to stop either with $S' = \emptyset$ (in which case $H = \{\underline{0}\}$) or with S' a linearly independent set spanning H, and hence a basis for H.

Bases for Nul A and Col A

We already have seen how to find a basis for Nul A: row reduce A to obtain a matrix in reduced row echelon form and use this to express the null space in vector parametric form. The vectors appearing will be the basis for Nul A.

Example: Let

$$B = \begin{pmatrix} 1 & 3 & 0 & 2 & 9 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Find a basis for Nul B.

For the same matrix B, find a basis for Col B.

Fact: If $A \sim B$, then the linear dependencies of the columns of A are exactly the same as the linear dependencies of the columns of B.

THEOREM 6. *The pivot columns of a matrix A form a basis for Col A.*

Note: A basis is

- A spanning set which is as small as possible
- A linearly independent set which is as big as possible

Example: which of the following sets of vectors form a basis for $\mathbb{R}^3$.

(1) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right\}$

(2) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

(3) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

4.4. Coordinate Systems

THEOREM 8 (Unique Representation Theorem). *Let V be a vector space, and let $\mathcal{B} = \{\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_n\}$ be a basis for V. Then for every $\underline{v} \in V$, there is a unique vector $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ so that*

$$\underline{v} = x_1\underline{b}_1 + x_2\underline{b}_2 + \cdots + x_n\underline{b}_n.$$

Proof: Since $\mathcal{B}$ is a basis, we know that there is at least one representation of this form (since $\underline{v} \in V$ and $\mathcal{B}$ spans V, $\underline{v}$ is a linear combination of the elements of $\mathcal{B}$.

So, we just have to show that there is only one such representation: this will follow from the fact that $\mathcal{B}$ is linearly independent.

Suppose that we also have

$$\underline{v} = y_1\underline{b}_1 + y_2\underline{b}_2 + \cdots + y_n\underline{b}_n.$$

We will show that $x_1 = y_1, x_2 = y_2, \ldots$. Subtracting the two expressions for $\underline{v}$, we obtain

$$\underline{0} = (x_1 - y_1)\underline{b}_1 + (x_2 - y_2)\underline{b}_2 + \cdots + (x_n - y_n)\underline{b}_n.$$

but since $\mathcal{B}$ is linearly independent, this implies that each of the values $x_1 - y_1$, $x_2 - y_2$, etc. must be 0. Hence $x_i = y_i$ as claimed.