These are brief notes for the lecture on Monday October 18, 2010: they are not complete, but they are a guide to what I want to say today. They are not guaranteed to be correct.

4.4. Coordinate Systems

Suppose that we are given a basis $\mathcal{B} = \{\underline{b}_1, \ldots, \underline{b}_n\}$ for the vector space $\mathbb{R}^n$. Then the vector equation

$$x_1\underline{b}_1 + x_2\underline{b}_2 + \cdots + x_p\underline{b}_n = \underline{v}$$

corresponds to the matrix equation

$$\underline{v} = P_{\mathcal{B}}\underline{x}.$$

But the values $x_1, x_2, \ldots, x_n$ in $\underline{x}$ are precisely the co-ordinates of $\underline{v}$ with respect to the basis $\mathcal{B}$. Hence

$$\underline{v} = P_{\mathcal{B}}[\underline{v}]_{\mathcal{B}}.$$

DEFINITION. *$P_{\mathcal{B}}$ is called the* change of co-ordinates *matrix.*

Now, since every vector is expressible uniquely, it means that the equation is solveable for every $\underline{v}$, and so $P_{\mathcal{B}}$ is invertible, and then

$$P_{\mathcal{B}}^{-1}\underline{v} = P_{\mathcal{B}}^{-1}P_{\mathcal{B}}[\underline{v}]_{\mathcal{B}} = [\underline{v}]_{\mathcal{B}}$$

that is,

$$[\underline{v}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\underline{v}.$$

So we can find the co-ordinates $[\underline{v}]_{\mathcal{B}}$ with respect to $\mathcal{B}$ of the vector $\underline{v}$ by multiplying it by $P_{\mathcal{B}}^{-1}$.

THEOREM 8. *Let $\mathcal{B} = \{\underline{b}_1, \ldots, \underline{b}_n\}$ be a basis for a vector space V. Let $T : V \to \mathbb{R}^n$ be defined by*

$$T(\underline{v}) = [\underline{v}]_{\mathcal{B}}.$$

Then T is a one-to-one *linear transformation* onto *$\mathbb{R}^n$.*

Note: The number of vectors in the basis is equal to n, the dimension of $\mathbb{R}^n$, the space we are mapping to.

We say that V is isomorphic to $\mathbb{R}^n$ (isomorphic meaning "same shape" or "same form"), which we write as $V \simeq \mathbb{R}^n$.

Corollary. *For any vectors* $\underline{v}, \underline{v}_1, \ldots, \underline{v}_k \in V$,

$$[\underline{v}]_{\mathcal{B}} = \underline{0} \qquad \iff \qquad \underline{v} = \underline{0}$$

and

$$[c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + \underline{v}_k]_{\mathcal{B}} = c_1[\underline{v}_1]_{\mathcal{B}} + c_2[\underline{v}_2]_{\mathcal{B}} + \cdots + c_k[\underline{v}_k]_{\mathcal{B}}.$$

4.5. The Dimension of a Vector Space

Theorem 10. *If a vector space* V *has a basis* $\mathcal{B} = \{\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_n\}$ *of cardinality* n*, then any subset of* V *with more than* n *vectors in is linearly dependent.*

Proof: Suppose that $p > n$ and $\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_p\}$ is a set of vectors in V. Then the coordinate vectors $[\underline{u}_1]_{\mathcal{B}}, [\underline{u}_2]_{\mathcal{B}}, \ldots, [\underline{u}_p]_{\mathcal{B}}$ form a linearly dependent set of vectors in $\mathbb{R}^n$ since $p > n$ and there are p of them.

Hence we can find scalars $c_1, c_2, \ldots, c_p$, not all zero, so that

$$c_1[\underline{u}_1]_{\mathcal{B}} + c_2[\underline{u}_2]_{\mathcal{B}} + \cdots + c_p[\underline{u}_p]_{\mathcal{B}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{(the zero vector in } \mathbb{R}^n\text{)}$$

Since the coordinate mapping is a linear transformation,

$$[c_1\underline{u}_1 + c_2\underline{u}_2 + \cdots + c_p\underline{u}_p]_{\mathcal{B}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

But since the coordinate mapping is one-to-one, this means that $c_1\underline{u}_1 + c_2\underline{u}_2 + \cdots + c_p\underline{u}_p = \underline{0}$, and since not all of the c_i are zero, the vectors are linearly dependent.

Theorem 11. *If* V *is a vector space with a basis of size* n*, then every basis for* V *has exactly* n *vectors.*

Proof: Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases having n and p vectors respectively. We will show that $n = p$. First, since $\mathcal{B}_1$ is a basis, and $\mathcal{B}_2$ is linearly independent, from the previous theorem we know that $p \leq n$. Similarly, since $\mathcal{B}_2$ is a basis, and $\mathcal{B}_1$ is linearly dependent, $n \leq p$. Thus $p \leq n \leq p$ and we see that $p = n$. □

Recall that if V is spanned by a finite set, then by repeatedly discarding vectors which are part of a non-trivial linear combination giving zero, we can find a basis for V. This theorem says that every basis must have the same number of vectors in it.

Definition. *If* V *is spanned by a finite set, then* V *is said to be* finite dimensional*, and the* dimension *of* V*, written as* $\dim V$*, is the number of vectors in a basis for* V*. The dimension of the zero vector space* $\{\underline{0}\}$ *is defined to be zero. If* V *is not spanned by a finite set, then* V *is said to be infinite dimensional.*

Example

$$\dim \mathbb{R}^n =$$

$$\dim \mathbb{P}_n =$$

$$\dim \mathbb{P} \ =$$

Example: Find the dimension of the subspace

$$H = \left\{ \begin{pmatrix} a+4b+c+2d \\ a+2b+d \\ a+5b+c+3d \\ b+d \end{pmatrix} \ : \ a,b,c,d \in \mathbb{R} \right\}$$

We now show that subspaces of finite dimensional vector spaces are also finite dimensional, and that we can build a basis in a natural way.

THEOREM 11. *If VR is a finite-dimensional vector space, and if $H < V$, then any linearly independent set $S \subset H$ can be expanded to a basis for H and*

$$\dim(H) \leq \dim(V).$$

Proof: The key idea is that if $S = \{\underline{u}_1, \ldots, \underline{u}_k\}$ and if S doesn't span H, then there is a vector $\underline{u}_{k+1} \notin \text{Span}(S)$, so that $\{\underline{u}_1, \ldots, \underline{u}_k, \underline{u}_{k+1}\}$ is still linearly independent. We continue enlarging S as long as it doesn't span H. This process has to stop, since the number of vectors we get can't exceed the dimension of V. Since S is then a basis for H, and the number of vectors in S is at most the dimension of V, we have $\dim(H) \leq \dim(V)$.

If we know the dimension of V, then finding a basis can be made somewhat simpler:

THEOREM 12. *Suppose that V is a p-dimensional vector space. Then*

(1) *Any linearly independent set of p vectors in V is a basis for V.*

(2) *Any set of p vectors which spans V is a basis for V.*

Proof *First, any linearly independent set can be extended to a basis. But a basis has to have p vectors in, hence it can't be any bigger. Hence the linearly independent set must already be a basis.*
Secondly, any spanning set contains a basis. But a basis has to have p vectors in, and there is only one subset of the spanning set having p vectors in, namely the whole set. Hence the spanning set must already be a basis.

The dimensions of Nul(A) and Col(A)

$$\dim(Col(A)) = \textit{ number of pivots in } A$$
$$\dim(Nul(A)) = \textit{ number of free variables in rref of } A$$

Note: the dimension of the null space is the number of columns of A minus the number of pivots.

Hence we have

$$\dim(\text{Col}(A)) + \dim(\text{Nul}(A)) = \text{ number of columns of } A.$$