These are brief notes for the lecture on Friday November 13, and Monday November 16, 2009: they are not complete, but they are a guide to what I want to say on those days. They are guaranteed to be incorrect.

6.2. Orthogonal Sets

THEOREM 5. *Let $W < \mathbb{R}^n$, and let $S = \{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_k\}$ be an orthogonal basis for W. Then if the vector $\underline{y}$ in W is given in terms of the basis S by*

$$\underline{y} = c_1\underline{u}_1 + c_2\underline{u}_2 + \cdots + c_k\underline{u}_k$$

then

$$c_j = \frac{\underline{y} \cdot \underline{u}_j}{\underline{u}_j \cdot \underline{u}_j}$$

Proof: Proof strategy: we know that every vector in W has a unique representation as a linear combination of vectors in S, since S is a basis for W. If we take the inner product of $\underline{y}$ with $\underline{u}_j$, then since W is orthogonal, all the inner products vanish except for the terms in the theorem.

Geometric Interpretation of Theorem 5: Let $\{\underline{u}_1, \underline{u}_2\}$ be an orthogonal basis for $\mathbb{R}^2$. Put

$$\hat{\underline{y}}_1 = \frac{\underline{y} \cdot \underline{u}_1}{\|\underline{u}_1\|^2}\underline{u}_1 = \text{Proj}_{\underline{u}_1}(\underline{y})$$

$$\hat{\underline{y}}_2 = \frac{\underline{y} \cdot \underline{u}_2}{\|\underline{u}_2\|^2}\underline{u}_2 = \text{Proj}_{\underline{u}_2}(\underline{y})$$

Then

$$\underline{y} = \hat{\underline{y}}_1 + \hat{\underline{y}}_2.$$

Orthonormal Sets

DEFINITION. *A set* $\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_p$ *is called* orthonormal *if it is orthogonal and* $\|\underline{u}_i\| = 1$ *for* $1 \leq i \leq p$. *In this case, if* $W = Span(\underline{u}_1, \ldots, \underline{u}_p)$*, then the set is called an* orthonormal basis *for* W*.*

Example: The set $\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n$ is an orthonormal basis for $\mathbb{R}^n$.
Example: Show that the set $\underline{v}_1, \underline{v}_2, \underline{v}_3$ is an orthornormal basis for $\mathbb{R}^3$, where

$$\underline{v}_1 = \begin{pmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{pmatrix}, \qquad \underline{v}_2 = \begin{pmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}, \qquad \underline{v}_3 = \begin{pmatrix} 1/\sqrt{66} \\ 4/\sqrt{66} \\ -7/\sqrt{66} \end{pmatrix}.$$

THEOREM 6. *An* $m \times n$ *matrix* U *has orthonormal columns if and only if* $U^T U = I$*.*

Proof: proof strategy: interpret the entries of $U^T U$ in terms of inner products of the columns of U.

THEOREM 7. *Let U be an $m \times n$ matrix with orthonormal columns, and $\underline{x}, \underline{y} \in \mathbb{R}^n$. Then*

(1) $\|U\underline{x}\| = \|\underline{x}\|$.

(2) $(U\underline{x}) \cdot (U\underline{y}) = \underline{x} \cdot \underline{y}$.

(3) $(U\underline{x}) \cdot (U\underline{y}) = 0$ *if and only if* $\underline{x} \cdot \underline{y}$.

Note: (a) U preserves length
(b) U preserves orthonormality.
Proof: proof strategy: write $(U\underline{x}) \cdot (U\underline{y})$ as

$$(U\underline{x}) \cdot (U\underline{y}) = (U\underline{x})^T(U\underline{y}) = (\underline{x}^T U^T)(U\underline{y}) = \underline{x}^T(U^T U)\underline{y}.$$

Example: Let

$$U = \begin{pmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{pmatrix} \qquad \text{and} \qquad \underline{x} = \begin{pmatrix} \sqrt{2} \\ 3 \end{pmatrix}.$$

Note that U has orthonormal columns and that $\|U\underline{x}\| = \|\underline{x}\|$.

In the case where U is a an $n \times n$ matrix with orthonormal columns, we see that $U^T U = I$ so $U^T = U^{-1}$, so $UU^T = I$ as well: that is, U has orthonormal rows too!