Combinatorial Analyis **Fall 2010**

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3.1 Combinatorial Classes:

We can extend the cross product of two combinatorial classes to finitely many $A \times B \times \ldots \times C \times D$.

If A, B, $\ldots$, C, D respectively have generating functions f_a, f_b, $\ldots$, f_c, f_d, then $A \times B \times \ldots \times C \times D$ has generating function $f_a(x)\ f_b(x)\ \ldots f_c(x)\ f_d(x)$, where the weight is
$w(a, b, ..., c, d) = w_a(a) + w_b(b) + \ldots + w_c(c) + w_d(d)$.

3.1.1 Example

Let $A = \{0, 1\}$ with $w(0) = 0$ and $w(1) = 1$ so A has generating function $1 + x$.

Let A^n denote $\underbrace{A \times A \times \ldots \times A}_{n-times}$ which has generating function (1+x)^n.

A typical element of A^n is an n-tuple of 0's and 1's (a binary sequence). Equivalently, a sequence of 0's and 1's of length n. A sequence has weight k if it has exactly k 1's.

Bijection ϕ_n : {Set of binary strings of length n} $\longrightarrow$ {Set of subsets of {1,2, . . . ,n}}.

$|\phi_n(\sigma)| = w(\sigma)$ and $\phi_n = \{$ j | $\sigma(j) = 1$ } = set of positions in which σ has a 1 .

e.g. 0110110111 $\rightarrow \{2, 3, 5, 6, 8, 9, 10\}$

Corollary 3.1
Number of subsets of $\{1, 2, \ldots, n\}$ *of cardinality* k
= Number of binary strings of length n *with* k *1's*
= Number of binary strings of length n *with weight* k
$= \left[x^k\right] (1 + x)^n$

Define $n \geq k \geq 0$, *n, k integer,* $\binom{n}{k} = \left[x^k\right] (1 + x)^n$.

Corollary 3.2 $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ *Pascal's identity*

Proof:

$$\left[x^k\right](1+x)^n = \left[x^k\right](1+x)^{n-1} + x(1+x)^{n-1} = \left(\left[x^k\right](1+x)^{n-1}\right) + \left(\left[x^k\right]x(1+x)^{n-1}\right)$$
$$= \binom{n-1}{k} + \left[x^{k-1}\right](1+x)^{n-1} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

■

Proof: [Combinatorial]

Let $\binom{n}{k}$ be the number of k-subsets of $\{1,2,\ldots,n\} = |S_{n,k}|$ where $S_{n,k}$ is the set of k-subsets of $\{1,2,\ldots,n\}$. Let $T_{n,k}$ be the set of k-subsets of $\{1,2,\ldots,n\}$ which contain the element n.

$S_{n,k} = S_{m-1,k}\dot{\cup}T_{n,k}$ where $\dot{\cup}$ is the disjoint union, that is $S_{n-1,k} \cap T_{n,k} = \emptyset$.

Now, $|T_{n,k}| = |S_{n-1,k-1}|$, indeed there is a natural bijection (delete the element n). Hence $|S_{n,k}| = |S_{n-1,k}| + |S_{n-1,k-1}|$, and so

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

as required. ■

3.1.2 Sets of Binary Strings via regular languages:

Take a nice language S; denote by S^* the set of all words formed by concatenating a finite number of elements of S. "nice" means that each word is obtained uniquely .

Example: $S = \{0,1\}$. $S^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 110, 111, \ldots\}$, where ϵ is the empty string.

Bad Example: The set $\{0,1,01\}$ is not nice since the string 01 is obtained as 01 and also (0)(1), so not obtained uniquely.

Is the set $\{0,01\}$ nice? yes. We can represent it the following ways:

$$0^{a_1}\dot{\cup}0^{a_1}10^{a_2}\dot{\cup}0^{a_1}10^{a_2}10^{a_3}\dot{\cup}\cdots$$

$$0^{b_1}\dot{\cup}0^{b_1}(01)0^{b_2}\dot{\cup}0^{b_1}(01)0^{b_2}(01)0^{b_3}\dot{\cup}\cdots$$

Back to considering binary sequences. $S = \{0,1\}$, and

$$\begin{aligned} S^* &= \text{all binary sequences} \\ &= \{0\}^*\{1\{1\}^*0\{0\}^*\}^*\{1\}^* \\ &\quad \text{which we'll write} \\ &= 0^*(11^*00^*)^*1^* \end{aligned}$$

What about the set of binary strings without consecutive 1's?

$$\{\epsilon, 0, 1, 00, 01, 10, 000, 001, 010, 100, 101, \ldots\} = 0^*(100^*)^*(\epsilon \cup 1)$$

Let's see if we can compute the generating funciton for such strings with x marking length.

$$1 + 2x + 3x^3 + 5x^2 + \cdots$$

0^* has generating function $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$. 100^* has generating function $\frac{x^2}{1-x}$. So, $(100^*)^*$ has generating function

$$1 + \left(\frac{x^2}{1-x}\right) + \left(\frac{x^2}{1-x}\right)^2 + \left(\frac{x^2}{1-x}\right)^3 + \cdots = \frac{1}{1 - \frac{x^2}{1-x}}$$

$\epsilon \cup 1$ has generating function $1 + x$. So, $0^*(100^*)^*(\epsilon \cup 1)$ has generating function

$$\frac{1}{1-x} \cdot \frac{1}{1 - \frac{x^2}{1-x}} \cdot 1 + x = \frac{1+x}{1-x-x^2}.$$

How can we get the Fibonacci numbers from this???