Combinatorial Analyis **Fall 2010**

Lecture 4: August 25

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4.1 Counting Strings Without Consecutive Ones:

We saw last class that the set of binary strings can be be expressed as $0^*(11^*00^*)^*1^*$. Furthermore, we saw that strings without adjacent ones can be expressed as $0^*(100^*)^*(\epsilon \cup 1)$.

4.1.1 Generating function:

The generating function for 0^*, with x marking length is

$$1 + x + x^2 + ... = \frac{1}{1-x}.$$

The generating function for $\epsilon \cup 1$, with x marking length is

$$1 + x.$$

The generating function for 100^*, with x marking length is

$$x^2 + x^3 + x^4 + x^5 + ... = \frac{x^2}{1-x}.$$

The generating function for $(100^*)^*$, with x marking length is

$$1 + \frac{x^2}{1-x} + \left(\frac{x^2}{1-x}\right)^2 + \left(\frac{x^2}{1-x}\right)^3 + ... = \frac{1}{1-\frac{x^2}{1-x}}.$$

So the generating function for $0^*(100^*)^*(\epsilon \cup 1)$ is

$$\frac{1}{1-x} \cdot \frac{1}{1-\frac{x^2}{1-x}} \cdot (1+x) = \frac{1+x}{1-x-x^2}.$$

4.1.2 Deriving Sequence from Generating Function:

How do we get the Fibonacci numbers out of $\frac{1+x}{1-x-x^2}$?
Suppose $f(x) = f_0 + f_1 x + f_2 x^2 + ... = \sum_{n \geq 0} f_n x^n$, and $f(x) = \frac{1+x}{1-x-x^2}$, that is, $(1-x-x^2)f(x) = 1 + x$.
Then $[x^n](1-x-x^2)f(x) = [x^n](1+x) = \begin{cases} 1 & : n = 0, 1 \\ 0 & : n \geq 2 \end{cases}$

$$[x^n]f(x) - xf(x) - x^2 f(x) = ([x^n]f(x)) - ([x^{n-1}]f(x)) - ([x^{n-2}]f(x)) = f_n - f_{n-1} - f_{n-2}, \text{if } n \geq 2.$$

So, if $n \geq 2$, $f_n - f_{n-1} - f_{n-2} = 0$.
So $f_n = f_{n-1} + f_{n-2}$.
If $n = 1$, we get $f_1 - f_0 = 1$.
If $n = 0$, we get $f_0 = 1$.
So $f(x) = 1 + 2x + 3x^2 + 5x^3 + 8x^4 + ...$
And the f_n's are (when suitably indexed) the Fibonacci numbers.

4.1.3 Exercises:

Exercise TBHI: By considering strings of length n starting with 0, and strings of length n starting with 1, (without adjacent 1's) show combinatorially that $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

Exercise TBHI: Reverse the argument above, and use $f_0 = 1$ and $f_1 = 2$ to show $(1 - x - x^2)f(x) = 1 + x$ and hence $f(x) = \frac{1+x}{1-x-x^2}$.

Exercise TBHI: By considering strings (without adjacent 1's) of length 2n+1 (or, more generally, $m + n + 1$) and separating them according to whether the middle bit is 0 or 1, show combinatorially that $f_{2n+1} = f_n^2 + f_{n-1}^2$ ($f_{m+n+1} = f_m f_n + f_{m-1} f_{n-1}$, respectively).

4.1.4 Generating Function Growth:

How fast does f_n grow?

Note that since f_n is an increasing sequence,
$f_n < 2f_{n-1}$, if $n \geq 2$.
$f_n > 2f_{n-2}$, if $n \geq 2$.

Hence, since $f_1 = 2^1$, $f_n < 2^n$ for $n \geq 2$. And since $f_0 = (2^{1/2})^n$ for all $n \geq 1$. So for all $n \geq 0$, $(2^{1/2})^n \leq f_n \leq 2^n$.

4.1.5 Further Analysis of f_n:

If f_n were exactly $c\alpha^n$ (which it is not!) then we would have $c\alpha^n = c\alpha^{n-1} + c\alpha^{n-2}$.
So, $\alpha = 0$, or $c = 0$, or $\alpha^2 - \alpha - 1 = 0$.
$\alpha = \frac{1 \pm \sqrt{5}}{2} = 1.6...$ or $-.6...$

Now observe that if $\alpha_1 = \frac{1+\sqrt{5}}{2}$ and $\alpha_2 = \frac{1-\sqrt{5}}{2}$ then $g_n = c_1\alpha_1^n + c_2\alpha_2^n$ also satisfies $g_n = g_{n-1} + g_{n-2}$.
If we solve $c_1\alpha_1^0 + c_2\alpha_2^0 = 1$ and $c_1\alpha_1^1 + c_2\alpha_2^1 = 2$, then $g_n = c_1\alpha_1^n + c_2\alpha_2^n = f_n$ for all n by induction.
$\begin{pmatrix} 1 & 1 \\ \alpha_1 & \alpha_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
Since $\alpha_1 \neq \alpha_2$, $\begin{pmatrix} 1 & 1 \\ \alpha_1 & \alpha_2 \end{pmatrix}$ is invertible, and hence this system *does* have a solution:
$c_1 + c_2 = 1$
$\frac{c_1}{2} + \frac{c_1\sqrt{5}}{2} + \frac{c_2}{2} - \frac{c_2\sqrt{5}}{2} = 2$
$c_1 + c_2 + (c_1 - c_2)\sqrt{5} = 4$
$c_1 - c_2 = \frac{3}{\sqrt{5}}$
$c_1 = \frac{1}{2}\left(1 + \frac{3}{\sqrt{5}}\right) = \frac{1}{2}\left(1 + \frac{3\sqrt{5}}{5}\right)$.
$c_2 = \frac{1}{2}\left(1 - \frac{3}{\sqrt{5}}\right) = \frac{1}{2}\left(1 - \frac{3\sqrt{5}}{5}\right)$.

So, $f_n = \frac{1}{2}\left(1+\frac{3\sqrt{5}}{5}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1}{2}\left(1-\frac{3\sqrt{5}}{5}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$.
Note that $\left|\frac{1}{2}\left(1-\frac{3\sqrt{5}}{5}\right)\right| < \frac{1}{2}$
and $\left|\frac{1-\sqrt{5}}{2}\right| < 1$
so $\left|\frac{1}{2}\left(1-\frac{3\sqrt{5}}{5}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n\right| < \frac{1}{2}$.
And so f_n is the closest integer to $\frac{1}{2}\left(1-\frac{3\sqrt{5}}{5}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$.

4.1.6 Back to $f(x) = \frac{1+x}{1-x-x^2}$:

If we consider $\frac{1}{1-\beta x}$, with constant β, as a formal power series,
we have $\frac{1}{1-\beta x} = 1 + \beta x + \beta^2 x^2 + \beta^3 x^3 + ...$
So, if we could write $\frac{1+x}{1-x-x^2}$ as $\frac{c_1}{1-\alpha_1 x} + \frac{c_2}{1-\alpha_2 x}$ then $f_n = c_1\alpha_1^n + c_2\alpha_2^n$.

To find $c_1, c_2, \alpha_1, \alpha_2$:
we need $(1-\alpha_1 x)(1-\alpha_2 x) = 1 - x - x^2$
or $\alpha_1 + \alpha_2 = 1$, $\alpha_1\alpha_2 = -1$

$$\alpha_2 = \frac{-1}{\alpha_1}$$

$$\alpha_1 - \frac{1}{\alpha_1} = 1$$

$$\alpha_1^2 - 1 = \alpha_1$$

$$\alpha_1^2 - \alpha_1 - 1 = 0$$

$$\alpha_1 = \frac{1+\sqrt{5}}{2}, \alpha_2 = \frac{1-\sqrt{5}}{2}$$

$$\frac{1+x}{(1-\alpha_1 x)(1-\alpha_2 x)} = \frac{c_1}{1-\alpha_1 x} + \frac{c_2}{1-\alpha_2 x}$$

So,

$$c_1(1-\alpha_2 x) + c_2(1-\alpha_1 x) = 1 + x$$

$$c_1 + c_2 = 1$$

$$c_1\alpha_2 + c_2\alpha_1 = -2$$

Solve, and obtain the same values as before.