Combinatorial Analysis	**Fall 2010**

Lecture 5: August 27

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5.1 Bivariate Generating Functions

Still considering the set of binary strings without adjacent 1's: $0^*(100^*)(\epsilon v1)$.
Consider the generating function (denoted g.f.) for two variables

$$f(x,y) = \sum_{f_n,k} x^n y^k \tag{5.1}$$

Where $f(x,y) = \#$ of binary strings of length n with exactly k 1's (still without adjacent 1's), and both n,k are finite.

Exercise: Suppose that $\mathcal{A}$ and $\mathcal{B}$ are combinatorial classes, with each having weight w and length l defined. That is each object has a non-negative integer weight and non-negative integer length and $\mathcal{A}_{n,k} = \{\text{objects with weight } n \text{ and length } k\}$, $\mathcal{B}_{n,k} = \{\text{objects with weight } n \text{ and length } k\}$.
$\|\mathcal{A}_{n,k}\|$ finite for each pair of n, k
$\|\mathcal{B}_{n,k}\|$ finite for each pair of n, k
If $\mathcal{A}$ has bivariate generating function $f(x,y)$, $\mathcal{B}$ has bivariate generating function $g(x,y)$, then $\mathcal{A} \times \mathcal{B}$ with obvious weight and length $w_p = w_1 + w_2$ and $l_p = l_1 + l_2$. Has bivariate generating function $f(x,y)g(x,y)$.
Once this exercise is complete we can do the following:

0^* has g.f. : $\dfrac{1 + x^2 + x^3 + ...1}{1-x}$

100^* has g.f. : $\dfrac{x^2y + x^3y + x^4y + ... = x^2y}{1-x}$

$(100^*)^k$ has g.f. : $\left(\dfrac{x^2y}{1-x}\right)^k$

Since these sets of strings are all disjoint

$(100^*)^*$ has g.f. : $1 + \dfrac{x^2y}{1-x} + \dfrac{x^2y}{1-x}^2 + \dfrac{x^2y}{1-x}^3 + ... = \dfrac{1}{1 - \frac{x^2y}{1-x}}$

$(\epsilon v1)$ has g.f. : $1 + xy$

Hence $0^*(100^*)(\epsilon v1)$ has g.f. : $\dfrac{1}{1-x}\dfrac{1}{1-\frac{x^2y}{1-x}}(1+xy) = \dfrac{1+xy}{1-x-x^2y}$

How do we get coefficients out of this?

$$
\begin{aligned}
&[y^k]\frac{1+xy}{1-x-x^2y} \\
&= [y^k]\frac{1}{1-x}\frac{1}{1-\frac{x^2y}{1-x}} \\
&= \frac{1}{1-x}[y^k]\frac{1}{1-\frac{x^2y}{1-x}} \\
&= \left(\frac{1}{1-x}[y^k]\frac{1}{1-\frac{x^2y}{1-x}}\right) + \left(\frac{x}{1-x}[y^{k-1}]\frac{1}{1-\frac{x^2y}{1-x}}\right) \\
&= \frac{1}{1-x}\left(\frac{x^2}{1-x}\right)^k + \frac{x}{1-x}\left(\frac{x^2}{1-x}\right)^{k-1}
\end{aligned}
$$

As an aside, note that:

$$
\begin{aligned}
&[x^m]\frac{1}{1-x^k} \\
&= [x^m](1-x)^{-k} \\
&= \binom{-k}{m}(-1)^m
\end{aligned}
$$

Where we need to interpret $\binom{-k}{m}$ properly! That is

$$
\begin{aligned}
\binom{n}{m} &= \frac{n!}{m!(n-m)!} \\
&= \frac{n(n-1)...(n-m+1)}{m!}
\end{aligned}
$$

This expression makes sense even if n is not a positive integer.

Definition 5.1 *Let z be an object which can be added or multiplied (i.e. in some ring). Then:*

$$
\binom{z}{m} := \frac{z(z-1)...(z-m+1)}{m!}
$$

So, for example

$$
\begin{aligned}
&\binom{-k}{m}(-1)^{-1} \\
&= \frac{-k(-k-1)\dots(-k-m+1)}{m!}(-1)^m \\
&= \frac{k(k+1)\dots(k+m-1)}{m!} \\
&= \frac{(k+m-1)(k+m-2)\dots(k+1)k}{m!} \\
&= \frac{(k+m-1)!}{m!(k-1)!} \\
&= \binom{k+m-1}{m} \\
&= \binom{k+m-1}{k-1}
\end{aligned}
$$

Note:

$$
\begin{aligned}
&\frac{1}{(1-x)^2} \\
&= \frac{d}{dx}(\frac{1}{1-x}) \\
&= \frac{d}{dx}(1+x+x^2+\dots) \\
&= 1+2x+3x^2+\dots \Rightarrow [x^m]\frac{1}{(1-x)^2} \\
&= m+1
\end{aligned}
$$

Let's continue this:

$$
\begin{aligned}
&(\frac{d}{dx})^2\frac{1}{1-x} = \frac{2}{(1-x)^3} \\
&(\frac{d}{dx})^3\frac{1}{1-x} = \frac{(2)(3)}{(1-x)^4} \\
&(\frac{d}{dx})^n\frac{1}{1-x} = \frac{n!}{(1-x)^{n+1}} \\
&\text{so, } \frac{1}{(1-x)^k} = \frac{1}{(k-1)!}(\frac{d}{dx})^{k-1}\frac{1}{1-x} \\
&\text{but } (\frac{d}{dx})^{k-1}x^r = r(r-1)\cdots(r-k+2)x^{r-k+1}
\end{aligned}
$$

so,

$$\begin{aligned}\frac{1}{(k-1)!}(\frac{d}{dx})^{k-1}x^r &= \frac{r(r-1)\cdots(r-k+2)}{(k-1)!}x^{r-k+1}\\ &= \binom{r}{k-1}x^{r-k+1}\end{aligned}$$

so,

$$[x^m]\frac{1}{(1-x)^k} = \binom{k+m-1}{k-1}$$

Theorem 5.2

$$[x^n](1+x)^z = \binom{z}{n} where \binom{z}{n} = \frac{z(z-1)\cdots(z-m+1)}{n!} \tag{5.2}$$

Back to coefficients,

$$[x^n y^k]f(x,y) = [x^n](\frac{1}{1-x}(\frac{x^2}{1-x})^k + \frac{x}{1-x}(\frac{x^2}{1-x})^{k-1})$$

$$\begin{aligned}[x^n]\frac{x^2k}{(1-x)^{k+1}} &= [x^{n-2k}]\frac{1}{(1-x)^{k+1}}\\ &= \binom{n-2k+k}{k}\\ &= \binom{n-k}{k}\end{aligned}$$

$$\begin{aligned}[x^n]\frac{x^{2k-1}}{(1-x)^k} &= [x^{n-2k+1}]\frac{1}{(1-x)^k}\\ &= \binom{n-2k+k}{k-1}\\ &= \binom{n-k}{k-1}\end{aligned}$$

$$\begin{aligned}&f_{n,k} = \binom{n-k}{k} + \binom{n-k}{k-1}\\ &= \binom{n-k+1}{k}\\ &\Rightarrow F_n = f_{n,0} + f_{n,1} + f_{n,2} + \dots\end{aligned}$$

So this expresses Fibonacci numbers in terms of binomial terms

$$F_n = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-k+1}{k}$$

Exercise: Give a combinatorial proof that $f_{n,k} = \binom{n-k+1}{k}$