Combinatorial Analyis **Fall 2010**

Lecture 6: August 30

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6.1 The Binomial Theorem for the unusual exponents:

We know

$$(1-x)^{-1/2} = \sum_{n\geq 0} \binom{-1/2}{n}(-1)^n x^n.$$

where

$$\begin{aligned}(-1)^n\binom{-1/2}{n} &= \frac{(-1/2)(-2/3)\dots(-1/2-n+1)}{n!}(-1)^n \\ &= \frac{(1)(3)(5)\dots(2n-1)}{2^n n!} \\ &= \frac{(1)(2)(3)(4)(5)\dots(2n-1)(2n)}{2^n n!((2)(4)\dots(2n))} \\ &= \frac{(2n)!}{2^n(n!)2^n(n!)} \\ &= \binom{2n}{n}\frac{1}{2^{2n}}.\end{aligned}$$

Hence,

$$(1-4y)^{-1/2} = \sum_{n\geq 0}\binom{2n}{n} y^n.$$

Exercise: Thus,

$$(1-4y)^{-1/2}(1-4y)^{-1/2} = (1-4y)^{-1} = \sum_{n\geq 0} 4^n y^n$$

and hence

$$\sum_{k=0}\binom{2k}{k}\binom{2n-2k}{n-k} = 4^n.$$

For example, when $n = 4$, we get

$$\binom{0}{0}\binom{8}{4} + \binom{2}{1}\binom{6}{3} + \binom{4}{2}\binom{4}{2} + \binom{6}{3}\binom{2}{1} + \binom{8}{4}\binom{0}{0} = 256 = 4^4.$$

Find a combinatorial proof of this.
Exercise: Find a relation between

$$\binom{-1/3}{n} \text{ and } \binom{-2/3}{n}$$

and

$$\binom{-1/4}{n} \text{ and } \binom{-3/4}{n}.$$

Simplify
$$\binom{-1/6}{n}.$$
using potentially, $\binom{-2/6}{n}, \binom{-3/6}{n}, \binom{-4/6}{n}, \binom{-2/6}{n}$. How many do we need?
Parenthesis: Product with n factors has c_n different interpretations as iterated binary products. For example, $a_1a_2a_3$ is $(a_1a_2)a_3$ or $a_1(a_2a_3)$. Then we showed

$$c_n = \sum_{k=1}^{n-1} c_k c_{k-1}, \quad n > 1$$

where $c = 1$. Hence, we know $c_0 = 0, c_1 = 1, c_2 = 1$, and $c_3 = 2$. Let $C(x) = \sum_{n\geq 1} c_n x^n$. Then consider

$$\begin{aligned} C(x)^2 &= \sum_{k\geq 1} c_k x^k \sum_{l\geq 1} c_l x^l \\ &= \sum_{n\geq 2} x^n \sum_{k=1}^{n-1} c_k c_{n-k} \\ &= C(x) - x. \end{aligned}$$

So $C(x)$ satisfies $C(x)^2 - C(x) + x = 0$. Applying the quadratic formula, we obtain

$$C[x] = \frac{1 \pm \sqrt{1-4x}}{2}.$$

Now

$$(1-4x)^{\frac{1}{2}} = \sum_{n\geq 0} \binom{\frac{1}{2}}{n} (-4)^n x^n$$

Since $\binom{\frac{1}{2}}{0} = 1, \binom{\frac{1}{2}}{1} = \frac{1}{2}$, and

$$\begin{aligned} \binom{\frac{1}{2}}{n} &= \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)...(\frac{1}{2}-n+1)}{n!} \\ &= \frac{1\cdot 1\cdot 3\cdot ...\cdot(2n-3)}{2^n n!}(-1)^{n-1} \\ &= \frac{1\cdot 1\cdot 3\cdot ...\cdot(2n-3)\cdot(2n-1)\cdot 2\cdot 4\cdot ...\cdot(2n-2)\cdot(2n)}{2^n n!\cdot(2n-1)\cdot 2\cdot 4\cdot ...\cdot(2n-2)\cdot(2n)}(-1)^{n-1} \\ &= \frac{(2n)!}{2^n n!\cdot(2n-1)2^n n!}(-1)^{n-1} \\ &= \frac{(2n)!}{4^n\cdot(2n-1)n!n!}(-1)^{n-1} \\ &= \frac{(-1)^{n-1}}{4^n\cdot(2n-1)}\binom{2n}{n} \quad \text{for } n \geq 2, \end{aligned}$$

it follows that

$$\begin{aligned} (1-4x)^{\frac{1}{2}} &= 1 - 2x - \sum_{n\geq 2} \frac{(-1)^{n-1}}{4^n\cdot(2n-1)}\binom{2n}{n}(-4)^n x^n \\ &= 1 - 2x - 2x^2 - 4x^3 - 10x^4 - .... \end{aligned}$$

Thus

$$C[x] = \frac{1 \pm (1 - 2x - 2x^2 - 4x^3 - 10x^4 - ...)}{2}.$$

Note that here we need to choose the sign to ensure that $C[0] = 0$, so we obtain

$$\begin{aligned} C[x] &= \frac{1 - (1 - 2x - 2x^2 - 4x^3 - 10x^4 - ...)}{2} \\ &= x + x^2 + 2x^3 + 5x^4 + ... \end{aligned}$$

So for $n \geq 1$,

$$C_n = \frac{1}{2} \cdot \frac{1}{2n-1} \binom{2n}{n} = \frac{1}{4n-2} \binom{2n}{n}.$$

Question: How can we get C_n without using the generating function?