Lecture 8: September 3

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8.1 Power Series Inverse

Example:

$$(1+x)(1-x+x^2-x^3+x^4+...)$$

converges in the ring of formal power series. So we write

$$(1+x)^{-1} = 1-x+x^2-x^3+x^4+...$$

Since we are in a commutative ring (for now) so $f(x)g(x) = g(x)f(x)$. If $f(x)$ has an inverse, $g(x)$ say, that is $f(x)g(x) = 1$ then it is unique.

(Non-commutative version: if $f(x)$ has a right inverse $g(x)$ and a left inverse $h(x)$ then $g(x) = h(x)$.)

Proof:

$$h(x)f(x) = 1 \qquad\qquad f(x)g(x) = 1$$

$$g(x) = (h(x)f(x))\,g(x) = h(x)\,(f(x)g(x)) = h(x)$$

■

8.2 Which power series have an inverse?

Suppose $f(x)$ has coefficient in a commutative ring, R (e.g. $\mathbb{Z}$). When does there exist $g(x)$ power series over the same ring so that $f(x)g(x) = 1$?

$$\begin{aligned} f(x) &= f_0 + f_1x + f_2x^2 + \cdots \\ g(x) &= g_0 + g_1x + g_2x^2 + \cdots \\ fg &= (f_0 + f_1x + f_2x^2 + \cdots)(g_0 + g_1x + g_2x^2 + \cdots) \\ &= f_0g_0 + (f_1g_0 + f_0g_1)x + (f_2g_0 + f_1g_1 + f_0g_2)x^2 + \cdots \end{aligned}$$

8.2.1 Method 1

So, if $fg = 1$, we need to simultaneously satisfy

$$f_0g_0 = 1 \quad \Rightarrow \quad f_0 \text{ must be a unit in } \mathbb{R}\text{, that is } f_0^{-1} \text{ exists so in } R = \mathbb{Z}\text{, this means } f_0 = \pm 1$$

$$f_1g_0 + f_0g_1 = 0$$

$f_2g_0 + f_1g_1 + f_0g_2 = 0$

$$
\begin{array}{llll}
\text{Then} & f_1g_0 = -f_0g_1 & \iff & g_1 = -f_0^{-1}f_1g_0 \\
 & & \implies & f_1g_0 + f_0g_1 = 0. \\
\text{Then} & f_2g_0 + f_1g_1 + f_0g_2 = 0 & \iff & g_2 = -f_0^{-1}(f_2g_0 + f_1g_1) \\
 & f_3g_0 + f_2g_1 + f_1g_2 + f_0g_3 = 0 & \iff & g_3 = -f_0^{-1}(f_3g_0 + f_2g_1 + f_1g_2)
\end{array}
$$

So we are able to construct (and actually compute!) $g(x) = f(x)^{-1}$.

8.2.2 Method 2

Proof: Alternative proof: If f_0^{-1} exist, $f(x)^{-1}$ exists.

$$
\begin{aligned}
(1-y)^{-1} &= 1 + y + y^2 \\
f(x) &= f_0(1 + f_0^{-1}f_1x + f_0^{-1}f_2x^2 + \cdots) \\
&= f_0(1 - x(-f_0^{-1}f_1 - f_0^{-1}f_2x - f_0^{-1}f_3x^2 + \cdots)) \\
&= f_0(1 - xh(x)) \text{ where } h(x) = -f_0^{-1}f_1 - f_0^{-1}f_2x - f_0^{-1}f_3x^2 + \cdots) \\
f(x)^{-1} &= (1 - xh(x))^{-1} - f_0^{-1} \\
&= (1 + xh(x) + x^2h(x)^2 + x^3h(x)^3 + \cdots)f_0^{-1}
\end{aligned}
$$

which converges since $|xh(x)|_\mu < 1$. ■

Example: If $f(x) = 1 - x - x^{k+1}$,

$$f(x)^{-1} = \frac{1}{1 - x(1+x)^k} = \sum_{k=0}^{\infty} x^\ell(1 + x^k)^\ell$$

$$
\begin{aligned}
[x^n]f(x)^{-1} &= \sum_{\ell=0}^{\infty}[x^{n-\ell}](1 + x^k)^\ell && \text{Need: } n - \ell \geq 0, k|(n-\ell) \\
&= \sum_{t=0}^{\lfloor \frac{n}{k} \rfloor}[x^{kt}](1 + x^k)^{n-kt} && \text{Put: } kt = n - \ell, 0 \leq kt \leq n \\
&= \sum_{t=0}^{\lfloor \frac{n}{k} \rfloor}\binom{n-kt}{t}
\end{aligned}
$$

8.2.3 Method 3

Under certain circumstances the following is easy-ish to compute. Set $f(x) = f_0(x)$ and suppose$f(0) = 1$.

$$\frac{1}{f_0(x)} = \frac{f_0(-x)}{f_0(x)f_0(-x)}$$

Since $f_0(x)f_0(-x)$ is even, we can write if as $f_1(x^2)$, where perhaps we can compute $f_1(y)$.

$$\begin{aligned}
\frac{1}{f_0(x)} &= \frac{f_0(-x)}{f_1(x^2)} \\
&= \frac{f_0(-x)f_1(-x^2)}{f_1(x^2)f_1(-x^2)} \\
&= f_0(-x)f_1(-x^2)f_2(-x^4)f_3(-x^8)\cdots \\
\text{Since} \quad & f_0 = 1 + a_1x + a_2x^2 + \cdots \\
\Rightarrow \quad & f_0(x)f_0(-x) = (1 + a_1x + a_2x^2 + \cdots)(1 + a_1x + a_2x^2 + \cdots) \\
&= 1 + (2a_2 - a_1^2)x^2 + \cdots) \\
&= 1 + b_1^2 + b_2x^4 + \cdots \\
\Rightarrow \quad & f_k(y) = 1 + c_1y + c_2y^2 + \cdots \\
& f_k(x^{2^k}) = 1 - c_1x^{2^k} + c_2x^{2^{k+1}} + \cdots
\end{aligned}$$

So, $f_k(x^{2^k}) \to 1$ as $k \to \infty$.

Note: to compute all coefficients in $\frac{1}{f(x)}$ up to x^N requires the product of $f_k(-x^{2^k})$ up to $k \geq \log_2 N$ ($k = \lceil \log_2 N \rceil$) giving a product of $k+1$ terms. We can do this by using Fourier Transformations.

This is fast precisely when we can compute $f_k(y)$ efficiently.

Exercise : $f(x) = 1 = x^j$**. What are the** f_k**'s? And what does this method give us?**

8.2.4 What happens with $f_0 = 1 - x - x^2$?

$$\begin{aligned}
f_0(-x) &= (1 + x - x^2) \\
f_0(x)f_0(-x) &= (1 + x - x^2)(1 - x - x^2) \\
&= (1 - 3x^2 + x^4) \\
f_1(y) &= 1 - 3y + y^2 \\
f_1(y)f_1(-y) &= (1 - 3y + y^2)(1 + 3y + y^2) \\
&= (1 - 9y^2 + 2y^2 + y^4) \\
&= 1 - 7y^2 + y^4 \\
f_2(z) &= 1 - 7z + z^4 \\
f_k(y) &= (1 - a_ky + y^2) \\
f_k(-y) &= (1 + a_ky + y^2) \\
f_k(y)f_k(-y) &= 1 - (a_k^2 - 2)y^2 + y^4) \\
a_{k+1} &= a_k^2 - 2
\end{aligned}$$

and obtain a recurrence to obtain a_k's, and hence a factorization of $\frac{1}{1-x-x^2}$.