Combinatorial Analysis **Fall 2010**

Lecture 9: September 6

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9.1 Formal Power Series

What operations can we perform, and when, on formal power series?

Aside: A nonzero Laurent series is a power series of the form $\sum_{k=m}^{\infty} a_k x^k$
where $a_m \neq 0$ and m is allowed to be negative.

All the questions we ask below about power series can be asked about Laurent series.

Exercise : Do so, and answer.

Let $f(x)$, $g(x)$ be formal power series: $f(x) = \sum_{n\geq 0} f_n x^n$ and $\sum_{n\geq 0} g_n x^n$:

1. We can add $f(x) + g(x)$, and obtain a formal power series

$$[x^n](f(x) + g(x)) = ([x^n]f(x)) + ([x^n]g(x))$$

2. We can multiply $[x^n]f(x)g(x) = \sum_{k=0}^{n} f_k g_{n-k}$.
3. We can divide $f(x)$ by $g(x)$ when $f(x) = \sum_{n\geq k} f_n x^n$, $g(x) = \sum_{m\geq \ell} g_m x^m$ and $\frac{1}{g_\ell}$ exists in the ring of coefficients and $k \geq \ell$.
4. Compute $\frac{1}{g(x)}$: if and only if $g_0 = g(0)$ is invertible in our ring of coefficients.
5. Compute $log(f(x))$?
   Does $log(y)$ have a formal power series representation? No
   However, $log(1 + y)$ does. Note $log(1) = 0$ and $\frac{d}{dy} log(1 + y) = \frac{1}{1+y} = \sum_{n\geq 0} (-1)^n y^n$ together imply that $log(1 + y) = \sum_{n\geq 0} \frac{(-1)^n y^{n+1}}{n+1} = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \dots$

Hence, if we hope for consistency, we'll want $log(f(x)) = log(1 + (f(x) - 1)) = \sum_{n \geq 0} \frac{(-1)^n (f(x)-1)^{n+1}}{n+1}$,

$$\begin{aligned} \text{which converges as a power series} \quad &\Leftrightarrow \quad |(f(x)-1)^{n-1}|_u \to 0 \\ &\Leftrightarrow \quad |(f(x)-1)|_u \leq 1 \\ &\Leftrightarrow \quad f(x) = 1 + g(x) \text{ with } |g(x)|_u \leq 1 \\ &\Leftrightarrow \quad f(0) = 1 \end{aligned}$$

6. Compute $exp(f(x))$?
   $e^y = \sum_{n \geq 0} \frac{1}{n!} y^n$, so $e^{f(x)} = \sum_{n \geq 0} \frac{f(x)^n}{n!}$, provided it coverges $\Leftrightarrow |f(x)|_u < 1$, that is $f(0) = 0$.

7. Compute $f(g(x))$?
   $f(g(x)) = \sum_{n \geq 0} f_n (g(x))^n$ which converges $\Leftrightarrow |f_n(g(x))^n|_u \to 0$
   $\Leftrightarrow |g(x)|_u < 1$ or $f_n = 0$ for all but finitely many n
   $\Leftrightarrow g(0) = 0$ or $f(x)$ is polynomial.

8. Compute $f'(x)$? Always $f'(x) = \sum_{n \geq 1} n f_n x^{n-1}$

Exercise :

1. **Let $f(x), g(x)$ be power series: Show $\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$.**

2. **Suppose $f(u)$ is a power series in u, $g(x)$ is a power series in x, $f(g(x))$ is a power series, show $f'(g(x))$ exists and $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$.**

Supppose $f(x)$ is a power series, and the appropriate series to follow exist. What is the best way to compute the following?
$[x^n]f(x)^m$
$[x^n]f(x)^a, a \notin \mathbb{N}$
$[x^n]e^{f(x)}$
$[x^n]log(1 + f(x))$.
Does the answer change if we want a sincgle coefficient, or all coefficients upto $\mathbb{N}$?

Iterated Squaring: suppose we can multiply: then compute a^n, express n in binary as $b_0 + 2b_1 + 4b_2 + \ldots + 2^k b_k$.

Simple method:
Compute

$$\begin{aligned} a^2 &= a \cdot a \\ a^4 &= a^2 \cdot a^2 \\ a^8 &= a^4 \cdot a^4 \\ &\vdots \\ a^{2^k} &= a^{2^{k-1}} \cdot a^{2^{k-1}} \end{aligned}$$

Then compute $a^n = \prod_{j | b_j = 1} a^{2^j}$.

Other method:
$a^{19} = a^{16+2+1}$
$a \rightarrow a^2 \rightarrow a^4 \rightarrow a^8 \rightarrow a^9 = a^8 \cdot a \rightarrow a^18 \rightarrow a^19 = a^18 \cdot a$
The procedure is this:
Start with a. For each binary digit, square, and if the digit is 1, multiply by a.