Combinatorial Analysis **Fall 2010**

Lecture 11: September 10

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11.1 How to Compute $[x^n] f(g(x))$

Back to computing $[x^n] e^{f(x)}$ and more generally $[x^n] f(g(x))$. We want to find $[x^n] f(g(x))$, this is, of course

$$\frac{1}{n!}\left(\frac{d}{dx}\right)^n f(g(x))\Big|_{x=0}.$$

To put this in context: recall the rule for diffferentiating a product

$$\left(\frac{d}{dx}\right)^n f(x)\, g(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x)\, g^{(n-k)}(x).$$

Exercise : Prove this by induction on n.

Now, how do we differentiate $f(g(x))$?

Observation:

$$\begin{aligned}
n=0, \quad f(g(x)) &= f^{(0)}(g(x))\, g^{(1)}(x)^0, \quad \text{silly} \\
n=1, \quad \frac{d}{dx} f(g(x)) &= f^{(1)}(g(x))\, g^{(1)}(x). \\
n=2, \quad \left(\frac{d}{dx}\right)^2 f(g(x)) &= \frac{d}{dx}\left(f^{(1)}(g(x))\right) g^{(1)}(x) + f^{(1)}(g(x)) \frac{d}{dx}\left(g^{(1)}(x)\right) \\
&= f^{(2)}(g(x))\, g^{(1)}(x)^2 + f^{(1)}(g(x))\, g^{(2)}(x). \\
n=3, \quad \left(\frac{d}{dx}\right)^3 f(g(x)) &= f^{(3)}(g(x))\, g^{(1)}(x)^3 + f^{(2)}(g(x))\, 2g^{(1)}(x)\, g^{(2)}(x) + f^{(2)}(g(x))\, g^{(1)}(x)\, g^{(2)}(x) \\
&+ f^{(1)}(g(x))\, g^{(3)}(x) \\
&= f^{(3)}(g(x))\, g^{(1)}(x)^3 + 3f^{(2)}(g(x))\, g^{(1)}(x)\, g^{(2)}(x) + f^{(1)}(g(x))\, g^{(3)}(x).
\end{aligned}$$

Write $f^{(m)}$ for $f^{(m)}(g(x))$ and g_m for $g^{(m)}(x)$. Then $\left(\frac{d}{dx}\right)^3 f(g(x))$ can be rewritten as $f^{(3)}g_1^3 + 3f^{(2)}g_2g_1 + f^{(1)}g_3$.

<u>General term for n</u>: For some $p \leqslant n$

$$f^{(p)} g_{\lambda_1} g_{\lambda_2} \cdots g_{\lambda_n}$$

with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_p \geqslant 1$ positive integers and $\lambda_1 + \lambda_2 + \cdots + \lambda_p = n$.

Definition: An integer partition $\lambda \vdash n$ of n is a non-increasing sequence

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_p \geqslant 1$$

of positive integers such that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_p = n,$$

where p is the number of parts in the partition, $p = p(\lambda)$, and λ is the size of the largest part.

It appears then, that

$$\left(\frac{d}{dx}\right)^n f(g(x)) = \sum_{\lambda \vdash n} c_\lambda f^{(p(\lambda))} g_\lambda$$

where g_λ denotes $g^{(\lambda_1)} g^{(\lambda_2)} \cdots g^{(\lambda_p)}$.

Exercise : Prove this by induction on n.

Exercise : Given $\lambda \vdash n$, write c_λ as a sum over partitions of λ' of $n-1$. The sum will involve the $c_{\lambda'}$'s.

Hint: ($c_{2,1}$ will contribute to $c_{3,1}$, $c_{2,2}$, $c_{2,1,1}$. $c_{2,2}$ will contribute (twice) to $c_{3,2}$, and (once) to $c_{2,1,1}$. $c_{3,2}$ comes from $c_{2,2}$. $c_{3,2} = 2c_{2,2} + c_{3,1}$.)

Exercise : Compute enough c_λ's to conjecture and prove a formula. The formula will be a simple expression form involving factorials or other functions of λ_1, λ_2, $\cdots$, λ_p, $p(\lambda)$ and i_1, i_2, i_3, $\cdots$ where i_k is the number of parts of size exactly k.

11.2 Solving Some Special Cases with Exponential Generating Functions

Observation: Once we've done this in too much generality, we see that it would be simpler to consider

$$\left(\frac{d}{dx}\right)^n e^{g(x)} = \sum_{\lambda \vdash n} c_\lambda e^{g(\lambda)} g(\lambda).$$

This also gives quite a bit of information about c_λ when λ has restricted part size. e.g. If λ has part size $\leqslant 2$, choose $g(x)$ to be a poly of degree 2 so that $g^{(3)} = 0$.

Example: Observation: If we take $g(x) = e^x - 1$, then $g^{(k)}(x) = e^x - 1, \forall k \geq 1$. So

$$\left(\frac{d}{dx}\right)^n e^{e^x - 1} = \sum_{\lambda \vdash n} c_\lambda e^{e^x - 1} e^{P(\lambda)x}.$$

Setting $x = 0$ on both sides, we get

$$n![x^n]e^{e^x-1} = [\frac{x^n}{n!}]e^{e^x-1} = \sum_{\lambda \vdash n} c_\lambda 1.$$

So

$$\sum_{\lambda \vdash n} c_\lambda = [\frac{x^n}{n!}]e^{e^x-1}.$$

That is,

$$e^{e^x-1} = \sum_{n\geq 0} b_n[\frac{x^n}{n!}],$$

where $\sum_{\lambda \vdash n} c_\lambda = b_n$. This is the exponential function for $b_0, b_1, ...$

Now, by using the exponential function for $b_0, b_1, ...$, we have:

$$\begin{aligned}
\frac{d}{dx}(e^{e^x-1}) = e^{e^x-1}e^x. \quad &\Rightarrow \quad \sum_{n\geq 1} \frac{b_n x^{n-1}}{(n-1)!} = (\sum_{n\geq 0} \frac{b_n x^n}{n!})(\sum_{m\geq 0} \frac{b_m x^m}{m!}) = \sum_n \sum_{k=0}^{n} \frac{b_k x^k}{k!} \frac{x^{n-k}}{(n-k)!} \\
&\Rightarrow \quad [x^{n-1}](\sum_{n\geq 1} \frac{b_n x^{n-1}}{(n-1)!}) = [x^{n-1}](\sum_n \sum_{k=0}^{n} \frac{b_k x^k}{k!} \frac{x^{n-k}}{(n-k)!}) \\
&\Rightarrow \quad \frac{b_n}{(n-1)!} = \frac{1}{(n-1)!} \sum_{k=0}^{n-1} \binom{n-1}{k} b_k \\
&\Rightarrow \quad b_n = \sum_{k=0}^{n-1} \binom{n-1}{k} b_k.
\end{aligned}$$