Combinatorial Analysis **Fall 2010**

Lecture 12: September 13

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12.1 Integer Partitions and Compositions

A partition λ of an integer n is a non-increasing sequence $\lambda_1, \lambda_2, \ldots, \lambda_k$ of positive integers so that $\lambda_1 + \lambda_2 + \ldots + \lambda_k = n$. We write $\lambda \vdash n$.
A composition of n is a sum $r_1 + r_2 + \ldots + r_k = n$. Various conventions apply: usually the number of summands is fixed and zero values are allowed.

12.1.1 Composition of n into exactly k non-negative parts

We can consider the generating function approach. Consider the k-tuple $(r_1, r_2, \ldots, r_k)$ with

$$\begin{gathered} r_1 \in \mathbb{Z}, r_1 \geq 0 \\ r_2 \in \mathbb{Z}, r_2 \geq 0 \\ \vdots \\ r_k \in \mathbb{Z}, r_k \geq 0. \end{gathered}$$

The generating function for all k-tuples is

$$((1-x)^{-1})^k = (1-x)^{-k}.$$

So the numbers of compositions in k parts ≥ 0 is

$$[x^n](1-x)^{-k} = (-1)^n \binom{-k}{n} = \binom{n+k-1}{n} = \binom{n+k-1}{k-1}.$$

12.1.2 Composition of n into exactly k positive parts

The generating function is

$$\left(\frac{x}{1-x}\right)^k.$$

Hence,

$$\begin{aligned} [x^n]\frac{x^k}{(1-x)^k} &= [x^{n-k}]\frac{1}{(1-x)^k} \\ &= \binom{-k}{n-k}(-1)^{n-1} \\ &= \binom{n-1}{k-1}. \end{aligned}$$

Corollary 12.1 *The number of compositions of n into positive parts is*

$$\binom{n-1}{0} + \binom{n-1}{1} + \ldots + \binom{n-1}{n-1} = 2^{n-1}.$$

Description of why this is so:
Take a string of n ones. Choose a subset of the $(n-1)$ spaces between the ones in each in the subset and write a +. Read this as a compositions written as unary. For example, if $n = 12$, $n - 1 = 11$, and the subset is $\{2, 6, 7, 10\}$.

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Thus, we $3 + 3 + 1 + 3 + 2$. Clearly, we have p_n, the number of partitions of n into positive parts $\leq 2^{n-1}$, the number of compositions of n in to positive parts.

12.1.3 Generating functions for partitions

Alternating representations for partitions via numbers of parts of each size. For example, $7 + 5 + 5 + 5 + 4 + 3 + 3 + 3 + 1 + 1$ will be represented as an infinite sequence $(i_1, i_2, i_3, \ldots)$, i_j = number of parts of size j. So in this example, we have $(2, 0, 4, 1, 3, 0, 1, 0, 0, \ldots)$ (only finitely many nonzero a terms). Generating function for i_1 is $\frac{1}{1-x}$. Generating function for i_2 is $\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \ldots$.

$$\begin{aligned} i_3 &= \frac{1}{1-x^3} \\ i_4 &= \frac{1}{1-x^4} \\ &\vdots \\ i_j &= \frac{1}{1-x^j}. \end{aligned}$$

Hence, the generating function for all partitions is now

$$\prod_{j=1}^{\infty}(1-x^j)^{-1} = \prod_{j=1}^{\infty}(1 + x^j + x^{2j} + \ldots).$$

Note that the j^{th} term in the product $(1 + x^j + x^{2j} + ...) \to_u 1$ as $j \to \infty$ so the product converges.

$$\begin{aligned} p_0 &= 1 = \text{empty sum} \\ p_1 &= 1 = 1 \\ p_2 &= 2 = 2, 1+1 \\ p_3 &= 3 = 3, 2+1, 1+1+1 \\ p_4 &= 5 = 4, 3+1, 2+2, 2+1+1, 1+1+1+1 \\ p_5 &= 7 = 5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1 \end{aligned}$$

Exercise: Show $p_n \leq p_{n-1} + p_{n-2}$ for $n \geq 2$.

Hence these p_n are less than the Fib numbers for $n \geq 5$. In fact, p_n grows slower than α^n for all $\alpha > 1$. True growth rate is $\simeq \frac{C}{n} e^{\alpha\sqrt{n}}$.

How to compute p_n: let $p(x) = \prod_{j=1}^{\infty}(1-x^j)^{-1}$.

How can we use this to

1. compute explicit values?
2. obtain growth rates for p_n?

Well,

1. $$x\frac{d}{dx}p(x) = x\sum_{j=1}^{\infty}\Big(\prod_{k=1}^{\infty}(1-x^k)^{-1}\Big)\frac{jx^{j-1}}{1-x^j} = p(x)\sum_{j=1}^{\infty}\frac{jx^j}{(1-x^j)}$$

$$\begin{aligned}\frac{xp'(x)}{p(x)} = \sum_{j=1}^{\infty}\frac{jx^j}{1-x^j} &= \sum_{j=1}^{\infty} jx^j + jx^{2j} + jx^{3j} + ... \\ &= \sum_{n=1}^{\infty} x^n \sum_{j|n} j \\ &= \sum_{n=1}^{\infty}\sigma(n)x^n\end{aligned}$$

So, $xp'(x) = p(x)\sum_{n\geq 1}\sigma(n)x^n$. Which means $[x^n]xp'(x) = np_n = \sum_{m=1}^{n} p_{n-m}\sigma(m)$. Then,

$$\begin{aligned} p_0 &= 1 \\ 1p_1 &= 1 \\ 2p_2 &= p_0\sigma(2) + p_1\sigma(1) = 4 \implies p_2 = 2 \\ 3p_3 &= p_0\sigma(3) + p_1\sigma(2) + p_2\sigma(1) = 9 \implies p_3 = 3 \end{aligned}$$

As an aside to see why the coefficients are divisors of the exponent:

$$\begin{array}{llll} x + x^2 & +x^3 + x^4 & +x^5 + x^6 & +x^7 + x^8 \\ \quad + 2x^2 & \quad\quad + 2x^4 & \quad\quad + 2x^6 & \quad\quad + 2x^8 \\ & +3x^3 & \quad\quad + 3x^6 & \\ & \quad\quad + 4x^4 & & \quad\quad + 4x^8 \\ & & +5x^5 & \\ & & & +7x^7 \\ & & & \quad\quad + 8x^8 \end{array}$$

So, $x + (1+2)x^2 + (1+3)x^3 + (1+2+4)x^4 + (1+5)x^5 + (1+2+3+6)x^6 + ...$