Lecture 13: September 15

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13.1 Integer Partitions

Computing p(n) via $np_n = \sum_{0 \le k \le n-1} p_k \sigma(n-k)$ requires $\theta(n)$ multiplications for p_n and $\theta(n)$ additions, one division, and hence $\theta(n^2)$ additions and multiplications and $\theta(n)$ divisions to compute p_0, p_1, ..., p_n.

Aside: $f(n) = \theta(g(n))$ means $\exists c_1, c_2 > 0$ s.t. $c_1 g(n) \le |f(n)| \le c_2 g(n)$. Here g(n) will be a positive function.

Computing $\sigma(1)$, $\sigma(2)$, ..., $\sigma(n)$ can be done with $< n\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}\right)$ operations:

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for j from 1 to n    σ(j) = 1 end
for j from 2 to n
    for k from 1 to ⌊n/j⌋
        σ(jk) = σ(jk) + j
    end
end
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1	2	3	4	5	6	7	8	9						
1	1	1	1	1	1	1	1	1						
	3		3		3		3	1						
		4			6			4						
			7				7							
				6										
					12									
						8								
							15							
								13						

$n + \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor + \ldots + \lfloor \frac{n}{2} \rfloor < n\left(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n}\right) = nH_n$

$H_n = \log n + \gamma + O\left(\frac{1}{n}\right)$

So number of operations is $< n \log n + O(n)$ (incrementing by 1).

General: Suppose f_n is a non-negative sequence and that $f(x) = \sum_{n \ge 0} f_n x^n$ has radius of convergence R.

Then, for any $x \in (0, R)$, $f_n x^n \le f(x)$.

Hence $f_n \le \inf_{x \in (0,R)} \frac{f(x)}{x^n}$.

Hence, if we solve $\frac{f'(x)}{x^n} - \frac{nf(x)}{x^n} = 0$, that is, $x = \frac{nf(x)}{f'(x)}$ say for x^*, $f_n \le \frac{f(x^*)}{(x^*)^n}$.

Example: $f_n = \frac{1}{n!}$, $f(x) = e^x \Rightarrow f'(x) = e^x$, so $x^* = n$ and $\frac{1}{n!} \leq \frac{e^9}{n}$ or $n! \geq \left(\frac{n}{e}\right)^n$.

Truth: $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$ that is $\frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} \to 1$ as $n \to \infty$.

Fairly typical behavior: If f_n grows nicely, "smoothly", then we obtain $f_n < \frac{f(x^*)}{(x^*)^n}$ and the "truth" is $f_n \sim \frac{c}{n^\alpha} \frac{f(x^*)}{(x^*)^n}$

$p_n = [x^n] \prod_{k=1}^{\infty} (1 - x^k)^{-1} = [x^n] \prod_{k=1}^{n} (1 - x^k)^{-1}$

So we can obtain an upperbound for p_n by $p_n \leq \frac{\prod_{k=1}^{n} (1-x^k)^{-1}}{x^n}$.

Exercise: How good a bound can you use this to give? Hint: $p_n \leq \sum_{k=1}^{n} -\log(1 - x^k) - n \log x$

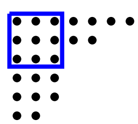
13.2 Restricted Partitions

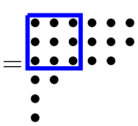
1. Partitions with all parts $\leq k$.
2. Partitions with at most k parts.
3. Partitions with all parts distinct.
4. Partitions with only odd parts.
5. Partitions in which parts differ by at least 2.

13.2.1 Ferrers' Diagram for a Partition

Define a partition $(\lambda_1 + \lambda_2 + \cdots + \lambda_k) \vdash n$.
Draw k lines opf dots, left justified, λ_j dots in the j^{th} line.

$7 + 5 + 3 + 3 + 3 + 2 = 23$

$\lambda =$ 

$\lambda' =$ 

$6 + 6 + 5 + 2 + 2 + 1 + 1 = 23$

There is a natural involution on the set of partitions of n, $\lambda \mapsto \lambda'$.
The conjugate, λ' of λ is the partition whose Ferrers' diagram is the transpose of that of λ.
Exercise: Using the $(i_1, i_2, \cdots)$ descriptions of λ, express λ'.

Definition 13.1 *The Durfee square of a Ferrers' diagram is the largest square which fits entirely inside the Ferrers' diagram.*
A Durfee square has size k if

$$\begin{aligned} \lambda_k &\geq k \\ \lambda_{k+1} &< k + 1 \end{aligned}$$

Observation: λ has all parts $\leq k \Longrightarrow \lambda'$ has at most k parts.

Corollary 13.2 *Fix k.*

$$\begin{aligned} & \textit{generating function for partitions of } n \textit{ into at most } k \textit{ parts} \\ = \; & \textit{generating function for partitions of } n \textit{ into parts of size at most } k \\ = \; & \prod_{j=1}^{k} \left(1 - x^j\right)^{-1} \end{aligned}$$

Corollary 13.3

$$\prod_{j=1}^{\infty} \left(1 - x^j\right)^{-1} \quad = \quad \sum_{k=0}^{\infty} x^{k^2} \prod_{j=1}^{k} \left(1 - x^j\right)^{-2}$$