Combinatorial Analyis **Fall 2010**

Lecture 15: September 22

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We can view a partition of n as a multiset $\{\lambda_1, \ldots, \lambda_n\}$ of elements choose from $\{1, 2, \ldots\}$ so that $\lambda_1, + \ldots + \lambda_k = n$ or better $w(\lambda_1) + \ldots w(\lambda_k) = n$. Multisets or unordered: this corresponds to the fact that we have a canonical way to write λ namely in non-increasing order.
We can regard the parts as connected objects which there are exactly on for each weight (length) and a partition is then a multiset of connected objects.

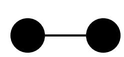
15.1 Unlabeled Trees and Forests

Trees = a connected acyclic graph on n vertices.
connected $\Rightarrow$ nonempty
Number of trees on n vertices: $t_n = \sum_{k=1}^{\infty} t_k x^k$

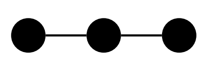
$$n = 0, t_n = 0$$

$$n = 1, t_n = 1$$

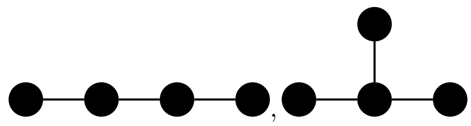
$$n = 2, t_n = 1$$



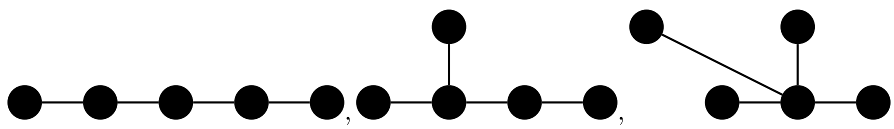
$$n = 3, t_n = 1$$



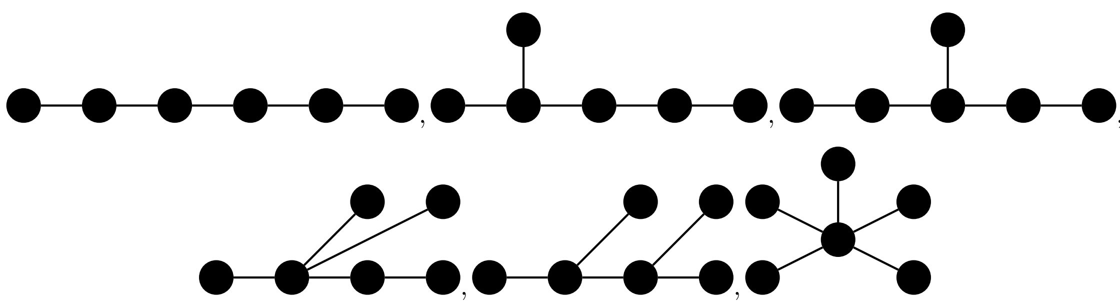
$$n = 4, t_n = 2$$



$n = 5, t_n = 3$



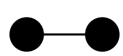
$n = 6, t_n = 6$



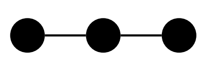
A forest is a multiset of trees. If we let f_n = the number of forests on n vertices, then how can we compute f_n?
List all trees:

T_1

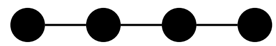
T_2



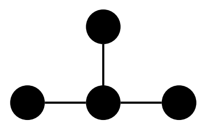
T_3



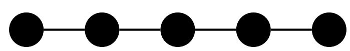
T_4



T_5



T_6



$\vdots$

Give a list $(i_1, \ldots)$ of the number of copies of each tree. i_j = number of copies of T_j. Then generating function then becomes

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} f_n x^n \\
&= (1 - x^{w(T_1)})^{-1} \ldots (1 - x^{w(T_n)})^{-1} \\
&= \prod_{j=1}^{\infty} (1 - x^{w(T_j)})^{-1}.
\end{aligned}$$

Grouping together by weight we see that we get

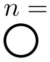
$$f(x) = \prod_{m=1}^{\infty} (1 - x^m)^{-t_m}.$$

We want to relate $f(x)$ and $t(x)$:

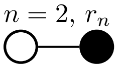
$$\begin{aligned} f(x) &= \exp(-\sum_{m\geq 1} t_m \log(1-x^m)) \\ &= \exp(\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\frac{t_m x^{mj}}{j}) \\ &= \exp(\sum_{j=1}^{\infty}\frac{1}{j}\sum_{m=1}^{\infty} t_m x^{mj}) \\ &= \exp(\sum_{j=1}^{\infty}\frac{1}{j}\sum_{m=1}^{\infty} t_m (x^j)^m) \\ &= \exp(\sum_{j=1}^{\infty}\frac{t(x^j)}{j}. \end{aligned}$$

Exactly the same method works and the same result holds true when $t(x)$ is the generating function for a set of connected objects and $f(x)$ is the generating function for multisets of connected objects.
Example: Graphs on n vertices, connected graphs on n vertices.
Example: Rooted unlabelled trees, rooted forests.

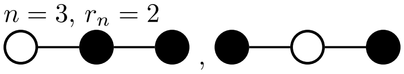
$n = 1$, $r_n = 1$



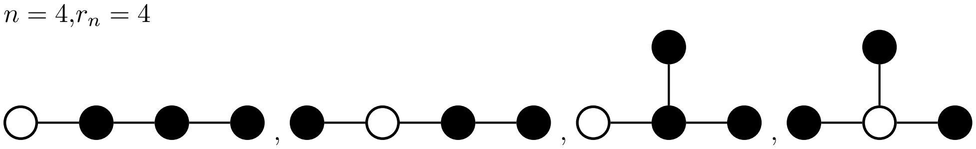
$n = 2$, $r_n = 1$



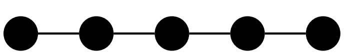
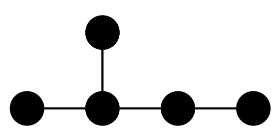
$n = 3$, $r_n = 2$

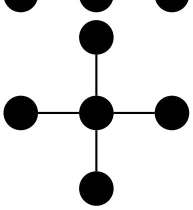


$n = 4$,$r_n = 4$

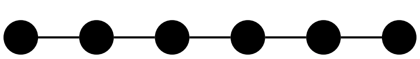
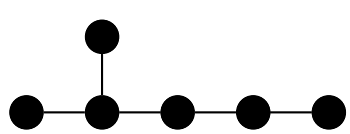


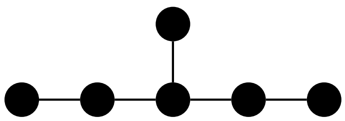
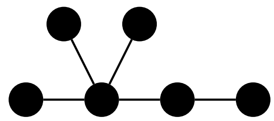
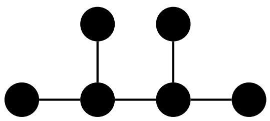
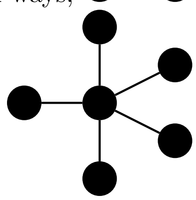
$n = 5$, $r_n = 9$


3 ways to root this,

4 ways to root this


2 ways to choose a root

$n = 6$, $r_n = 20$


3 ways,

5 ways,


4 ways,

4 ways,

2 ways,

2 ways.

Let $r(x) = \sum_{k\geq 1} r_k x^k$

$g(x) = \sum_{n\geq 0} g_n x^n$, g_n = number of rooted forests on n vertices

$g(x) = exp\{\sum \frac{1}{j} r(x^j)\}$

number of rooted forests on n vertices = number of rooted trees on n+1 vertices (bijection: chop off root)

$\Rightarrow r(x) = xg(x)$

so $r(x)$ satisfies the functional equation $r(x) = xexp\{\sum_{j=1}^{\infty} \frac{1}{j} r(x^j)\}$

Excercise: Use this functional equation to obtain enough information (a recurrence) to compute $r_1, r_2, \cdots, r_{10}$

Sets of connected objects:

$$
\begin{aligned}
h(x) = \sum_{n\geq 0} h_n x^n &= \prod_{j=1}^{\infty} (1+x^j)^{t_j} \\
&= exp\{\sum_{m=1}^{\infty} t_m log(1+x^m)\} \\
&= exp\{\sum_{m=1}^{\infty} t_m \sum_{j=1}^{\infty} (-1)^{j+1} \frac{x^{mj}}{j}\} \\
&= exp\{-\sum_{j=1}^{\infty} \frac{(-1)^j}{j} \sum_{m=1}^{\infty} t_m x^{mj}\} \\
&= exp\{-\sum_{j=1}^{\infty} \frac{(-1)^j}{j} t(x^j)\} \\
&= exp\{t(x) - \frac{1}{2}(x^2) + \frac{1}{3} t(x^3) - \cdots\}
\end{aligned}
$$