Combinatorial Analysis	**Fall 2010**

Lecture 17: September 24

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17.1 Another Construction

Again, we'll have a family of nontrivial objects $\mathcal{B}$ with generating function $B(x)$. (The $\mathcal{B}$ will play the role of connected graphs or partitions etc.) Then we can consider all sequences of m elements of $\mathcal{B}$

$$(b_1, b_2, b_2, \ldots, b_m)$$

with repetition allowed. This has generating function $B(x)^m$. Since all the elements of $\mathcal{B}$ are nontrivial, i.e. have positive weight, $B(0) = 0$, so $|B(x)|_u < 1$, so $\sum_{m \geq 0} B(x)^m$ converges and hence the set of all finite sequences of elements of B has generating function

$$\frac{1}{1 - B(x)}.$$

And, if we wanted to enumerate not just by weight, but by length of sequence as well, we could consider

$$\sum_{m \geq 0} y^m B(x)^m = \frac{1}{1 - yB(x)}.$$

Example: 01 strings.

17.1.1 New construction

Now we will introduce a new construction: Say that two sequences $(b_1, b_2, \ldots, b_m)$, $(c_1, c_2, \ldots, c_m)$ are equivalent, $\sim$, if one is a cyclic rotation of the other. So, we have

$$(b_1, b_2, b_3, \ldots, b_m) \sim (b_2, b_3, \ldots, b_m, b_1) \sim (b_3, b_4 \ldots, b_m, b_1, b_2) \sim \cdots \sim (b_m, b_1, b_2, \ldots, b_{m-1})$$

Now, consider the set

$$\mathcal{C}_m = \mathcal{B}^m / \sim$$

of necklaces, that is, equivalence classes of sequences of length m under $\sim$. (Note: There is another term, bracelets, which is similar, only reflections cause elements to be in the same equivalence class as well.)

Example: How many necklaces of 0s and 1s are there?

m	strings	number
0	empty string	1
1	$0, 1$	2
2	$00, 01, 11$	3
3	$000, 001, 011, 111$	4
4	$0000, 0001, 0011, 0101, 0111, 1111$	6
5	$00000, 00001, 00011, 00101,$ $00111, 01011, 01111, 11111$	8
6	$000000, 000001, 000011, 000101,$ $001001, 000111, 001011, 010101,$ $010011, 110110, 111010, 111100,$ $111110, 111111$	14

Turns out that the generating function for necklaces of all lengths is

$$N(x) = \sum \frac{-\varphi(k)}{k} \log\left(1 - B(x^k)\right).$$

This is also known as, or comes from, the Polya Enumeration Theory (PET).

17.2 Counting Set Partitions

Definition 17.1 *A set partition of S with k parts is a set*

$$\{c_1, c_2, \ldots, c_k\}$$

of nonempty disjoint sets whose union is S. That is,

1. $c_i \neq \emptyset$ *for all* i
2. $c_i \cap c_j = \emptyset$ *for all* $i < j$
3. $\bigcup_j c_j = S$.

17.2.1 Question:

How many set partitions of $\{1, 2, \ldots, n\}$ are there? Call this number B_n.
Generating functions here need to be different (or at least it is helpful if they are since the number of set partitions of n grows much faster than the number of subsets of $\{1, 2, \ldots, n\}$).
We'll use the exponential generating function for the sequence $\{B_n\}_{n\geq 0}$, namely

$$B(x) = \sum_{n\geq 0} B_n \frac{x^n}{n!}$$

so that

$$B_n = \left[\frac{x^n}{n!}\right] B(x) = n![x^n]B(x).$$

(This construction gives $B(x)$ a positive radius of convergence.)

17.2.2 What is $B(x)$?

n	set paritions	
0	$\{\}$	$B_0 = 1$
1	$\{\{1\}\}$	$B_1 = 1$
2	$\{\{1\},\{2\}\},\{\{1,2\}\}$	$B_2 = 2$
3	$\{\{1\},\{2\},\{3\}\},\{\{1,2\},\{3\}\},\{\{1,3\},\{2\}\},\{\{2,3\},\{1\}\},\{\{1,2,3\}\}$	$B_3 = 5$
4		$B_4 = 15$

Can we obtain a recurrence for B_n? If we collect/group the set partitions of $\{1, 2, \ldots, n_1\}$ according to the size of the part containing $n+1$ we can. Let k be the size of the part containing $n+1$, without counting $n+1$. Then we have

$$B_{n-1} = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}$$

Now notice something about multiplying generating exponential functions. Suppose

$$f(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} \qquad \text{and} \qquad g(x) = \sum_{n=0}^{\infty} g_n \frac{x^n}{n!}.$$

Then,

$$\begin{aligned} f(x)g(x) &= f(x) = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \frac{f_k}{k!} \frac{g_{n-k}}{(n-k)!} \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} f_k g_{n-k} \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n} \binom{n}{k} f_k g_{n-k}. \end{aligned}$$

So,

$$\left[\frac{x^n}{n!}\right] f(x)g(x) = \sum_{k=0}^{n} f_k g_{n-k} \binom{n}{k}.$$

So,

$$\begin{aligned} B_{n-1} &= \sum_{k=0}^{n} \binom{n}{k} \cdot 1 \cdot B_{n-k} \\ &= \left[\frac{x^n}{n!}\right] e^x B(x) \end{aligned}$$

Now,

$$B'(x) = \sum_{n \geq 1} \frac{n x^{n-1}}{n!} B(x)$$

so,

$$\begin{aligned} B_{n+1} &= \left[\tfrac{x^n}{n!}\right] B'(x) \\ \Rightarrow \quad \frac{B'(x)}{B(x)} &= e^x \\ \Rightarrow \quad \frac{d}{dx}\log(B(x)) &= e^x \\ \Rightarrow \quad \log(B(x)) &= e^x + c \\ \Rightarrow \quad B(x) &= e^{e^x + c} \\ \Rightarrow \quad B(x) &= e^{e^x - 1} \end{aligned}$$

The last implication comes from the idea that

$$e^{e^x + c} = \sum_{k=0}^{\infty} \frac{3^x + c)^k}{k!}$$

needs to converge as a power series. So we need

$$\begin{aligned} |e^x + c|_u &< 1 \\ e^0 + c &= 0 \\ c &= -1. \end{aligned}$$