Combinatorial Analyis **Fall 2010**

Lecture 18: September 27

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18.1 Set Partitions

$A(z)$ is the generating function for necklaces of objects from $\mathcal{B}$. $\mathcal{B}$ having generating fuction $B(z)$, then

$$A(z) = \sum_{k=1}^{\infty} \frac{-\varphi(k)}{k} \log\left(1 - B\left(z^k\right)\right).$$

Last time: Generating function for set partitions is

$$b(z) = \sum_{n\geq 0} b_n \frac{z^n}{n!} = e^{e^z - 1},$$

the exponential generating function for b_n. What does this tell us about, say, the growth rate of b_n?

$$\frac{b_n}{n!} \leq \frac{e^{e^x - 1}}{x^n}, \quad \forall x > 0,$$

minimmized when

$$\frac{e^{e^x-1}\cdot e^x}{x^n} - \frac{n \cdot e^{e^x - 1}}{x^{n+1}} = 0$$

i.e. $n = xe^x$.

Exercise (NTBHI): Learn about Lambert's W function.

How does xe^x behave for $x \in (0, \infty)$?

$\frac{d}{dx}(xe^x) = (x+1)e^x > 0$, $\forall x \in (0,1)$ and $xe^x|_{x=0} = 0$ and $\lim_{x\to\infty} xe^x = \infty$. Hence for any real $y \geq 0$, $y = xe^x$ has a unique solution x with $x \geq 0$.

How big should x be if $xe^x = n$? (n large)

x should be on the order of magnitude of $\log n$. $x = \log n$ is too big since $\log n \cdot e^{\log n} = n \cdot \log n > n$. Observe $x = \log\left(\frac{n}{\log n}\right) < \log n$:

$$\log\left(\frac{n}{\log n}\right) \cdot e^{\log\left(\frac{n}{\log n}\right)} = \frac{n\left(\log n - \log\log n\right)}{\log n} < n.$$

So it appears that $x = \log n - \log\log n$ is a better estimate. Set $x = \log n - \log\log n + c$, then

$$xe^x = \frac{\log n - \log\log n + c}{\log n} \cdot ne^c,$$

so

$$e^{-c} = 1 - \frac{\log\log n}{\log n} + \frac{c}{\log n}.$$

So, assuming c is $o(1)$, terms up to $O\left(c^2\right)$, give

$$1 - c = 1 - \frac{\log\log n}{\log n} + \frac{c}{\log n}.$$

So $c\left(\frac{\log n - 1}{\log n}\right) = \frac{\log\log n}{\log n}$, and hence $c = \frac{\log\log n}{\log n - 1}$. So our minimizing happens for

$$x \simeq \log n - \log\log n + \frac{\log\log n}{\log n - 1} + o(1).$$

Then, using, say $x = \log n - \log\log n$,

$$e^{e^x - 1} = e^{\frac{n}{\log n} - 1} = \left(e^{\frac{1}{\log n}}\right)^n \cdot \frac{1}{e},$$

$$x^n = \left(\log n\left(1 - \frac{\log\log n}{\log n}\right)\right)^n \simeq (\log n)^n e^{-\frac{n\log\log n}{\log n}}.$$

So

$$\frac{b_n}{n!} \leq e^{\left(\frac{1}{\log n} - \log\log n + \frac{\log\log n}{\log n}\right)^n} \cdot \frac{1}{e} \simeq \left(e^{-\log\log n}\right)^n < d^n$$

for any fixed $d > 0$.

So $\frac{b_n}{n!} \to 0$ very rapidly (but not as rapidly as $\frac{1}{n!}$).

$$n! \simeq \left(\frac{n}{e}\right)^n \sqrt{2\pi n} = e^{n\log n - n + \frac{1}{2}\log n + \cdots}$$

b_n grows like

$$e^{n\log n - n\log\log n - n + n\frac{1+\log\log n}{\log n} + \cdots}$$

Not very illuminating, it says more if we say $\log b_n \simeq n\log n - n\log\log n + \cdots$

If b_n doesn't have a simple growth rate, then we probably can't expect some of the following to either ...

Define $S(n,k)$ to be the number of set partitions of $\{1, 2, \cdots, n\}$ having exactly k parts.

$S(n,1) = 1$
$S(n,n) = 1$
$S(n,k) = \frac{1}{k!} \cdot \#$ onto functions from $\{1, 2, \cdots, n\}$ to $\{1, 2, \cdots, k\}$
$S(n,n-1) = \binom{n}{2}$
$S(n,2) = 2^{n-1} - 1$
$S(n,n-2) = ?$ part sizes possible: $2, 2, 1, \cdots, 1$ or $3, 1, 1, \cdots, 1$.

$$S(n,n-2) = \binom{n}{3} + \frac{1}{2}\binom{n}{2}\binom{n-2}{2} = \frac{n(n-1)(n-2)}{6} + \frac{n(n-1)(n-2)(n-3)}{8}.$$

This appears perhaps to give a little insight:

$$S(n,k) = \sum_{\substack{\lambda \vdash n \\ \lambda \text{ has } k \text{ parts}}} \binom{n}{\lambda_1, \lambda_2, \cdots, \lambda_k} \frac{1}{i_1! i_2! i_3! \cdots},$$

where $i_j = \#$ copies of j in λ.

Exercise: c_λ?

Easier approach to $S(n,k)$: $S(n+1,k) = S(n,k-1) + kS(n,k)$
compare to Pascal's identity $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

So, $S(1,1) = 1$
$S(2,1) = 1$
$S(2,2) = 1$

				k			
n	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	3	1				
4	1	7	6	1			
5	1	15	25	10	1		
6	1	31	90	65	15	1	

What about other similar recurrences?

$T(n+1,k) = f(n,k) \cdot T(n,k-1) + g(n,k) \cdot T(n,k)$ where $f(n,k)$ and $g(n,k)$ are nice simple functions.

e.g.

$T(n+1,k) = k \cdot T(n,k-1) + T(n,k)$?

$T(n+1,k) = T(n,k-1) + n \cdot T(n,k)$?

$T(n+1,k) = n \cdot T(n,k-1) + T(n,k)$?

$T(n+1,k) = T(n,k-1) + (n-k) \cdot T(n,k)$?

Exercise: Investigate these computationally. Are there any nice patterns? Do you ever get nice row sums? diagonal sums? (Due Next Friday on Oct. 8th)