Combinatorial Analysis **Fall 2010**

Lecture 21: October 04

Lecturer: Neil Calkin *Scribe: Grady Thomas and Honghai Xu*

Disclaimer: *These notes are intended for students in the class listed above: they are not guaranteed to be complete or even necessarily correct. They may only be redistributed with permission, which you may expect will be liberally granted. Ask first, please.*

21.1 What is the distribution of cycles in a random permutation?

Specifically if we pick a permutation of 1,2,...,n and a fixed integer k, what is the probability that there are exactly k fixed points?

P(π has no fixed points) $= \frac{D_n}{n!} \simeq \frac{1}{e}$
How many permutations have exactly one fixed point? $\binom{n}{1} D_{n-1}$
How many permutations have exactly k fixed points? $\binom{n}{k} D_{n-k}$
Denote, temporarily, $\lfloor x + \frac{1}{2} \rfloor$ to be the nearest integer to x
so,

$$\begin{aligned} P(\pi \text{ has exactly k fixed points}) &= \frac{1}{n!} \frac{n!}{k!(n-k)!} D_{n-k} \\ &= \frac{1}{k!} \frac{\lfloor \frac{(n-k)!}{e} + \frac{1}{2} \rfloor}{(n-k)!} \\ &\simeq \frac{1}{k!e} \\ &= \frac{e^{-1}}{k!} + error_{n,k}, \text{where } |error_{n,k}| < \frac{1}{k!(n-k+1)!} \end{aligned}$$

Asymptotically, the number of fixed points is almost Poisson, with mean 1.
So, expected number of fixed points is about $E[X]$ where $X \sim Poisson(1)$, so $E[X] = 1$

Can we count 2-cycles?
One Approach: Compute the probability that there are no 2-cycles. Prove some general abstract nonsense (here, Poisson paradigm) to show distribution is asymptotically Poisson, and deduce the mean.

Exponential generating function for permutations without 2-cycles is

$$\exp\{-log(1-x)-\frac{x^2}{2}\}=\frac{e^{\frac{-x^2}{2}}}{1-x}$$

We could, if we choose, write $e^{\frac{-x^2}{2}}=\sum\frac{x^{2k}}{k!}\frac{(-1)^k}{2^k}$ and compute exactly as we did with D_n.
Excercise: Compute as we did with D_n, get a similar "nearest integer" result.

Alternative Approach: Since $\frac{e^{\frac{-x^2}{2}}}{1-x}$ blows up at x=1 we know that if $\frac{e^{\frac{-x^2}{2}}}{1-x}=\sum_{n\geq 0}t_n\frac{x^n}{n!}$, then for any $|r|<1$, $\frac{r^n t_n}{n!}\to 0$ and for any $|R>1, \limsup_{n\to\infty}\frac{R^n t_n}{n!}\to\infty$

This suggest $t_n\approx cn!$.

Let's see if we can eliminate the singularity at $x=1$.
At $x=1$, $e^{-\frac{x^2}{2}}$ behaves like $e^{-\frac{1}{2}}$.
Near $x=1$, $\dfrac{e^{-\frac{x^2}{2}}}{1-x}$ behaves like $\dfrac{e^{-\frac{1}{2}}}{1-x}$.
So, consider $\dfrac{e^{-\frac{x^2}{2}}}{1-x}-\dfrac{e^{-\frac{1}{2}}}{1-x}$, say, for $x=1-\delta$.

$$\frac{e^{-\frac{x^2}{2}}}{1-x}-\frac{e^{-\frac{1}{2}}}{1-x}=\frac{e^{-\frac{(1-\delta)^2}{2}}-e^{-\frac{1}{2}}}{\delta}=\frac{e^{-\frac{1}{2}}}{\delta}\cdot(e^{\delta-\frac{\delta^2}{2}}-1)$$

As $\delta\to 0$, $(\dfrac{e^{\delta-\frac{\delta^2}{2}}-1}{\delta})\to 1$. Hence, $\dfrac{e^{-\frac{x^2}{2}}-e^{-\frac{1}{2}}}{1-x}$ has no singularities in C, and its coefficients approach 0 faster than ϵ^n for all $\epsilon>0\Rightarrow\dfrac{t^n}{n!}=e^{-\frac{1}{2}}+O(\epsilon^n)$, for any fixed $\epsilon>0$.
Therefore, the probability of an arbitrary permutation π has no two-cycles $\to e^{-\frac{1}{2}}$ as $n\to\infty$.

Similarly, the probability of an arbitrary permutation π has no k-cycles $\to e^{-\frac{1}{k}}$ as $n\to\infty$.
Exercise : Prove this proposition.

Hence a poisson paradigm Theorem would tell us that the number of k-cycles is poisson with mean $\approx\frac{1}{k}$ and hence the number of k-cycles is $\approx\frac{1}{k}$.

As a collary, the expected number of cycles in a random permutation on $\{1,2,...,n\}$ is

$$\sum_{k=1}^{n}E(\text{the number of k-cycles})\approx\sum_{k=1}^{n}\frac{1}{k}\approx H_n.$$

$$H_n=1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}\approx\log n+r+O(\frac{1}{n}),$$

where r is the Euler Mascheroni Constant.

Let's check the above formula. We notice that for $n = 1, 2, 3, 4$, the formula is more than just an approximation. The equality holds, as the following table shows.

n	H_n	permutations	E(number of cycles)
1	1	(1)	1
2	$1 + \frac{1}{2} = \frac{3}{2}$	(1)(2);(12)	$\frac{3}{2}$
3	$\frac{3}{2} + \frac{1}{3} = \frac{11}{6}$	(1)(2)(3);(1)(23);(2)(13);(3)(12);(123);(132)	$\frac{11}{6}$
4	$\frac{11}{6} + \frac{1}{4} = \frac{25}{12}$		$\frac{1\times4+6\times3+3\times2+8\times2+6\times1}{4+3+2+2+1} = \frac{25}{12}$

Table 21.1: Comparison of values of H_n and E(number of cycles)

Exercise : How far does this pattern continue?