Cancellation in Cyclic Consecutive Systems

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Abstract

We consider the structure and number of non-zero terms in the reliability polynomials for cyclic consecutive systems. We explain the large amount of cancellation, the fact that all but one of the coefficients are 0, 1 or -1, and show that the number of non-zero coefficients is asymptotic to α^k, where α is the largest root of $2 + x^r - x^{r+1} = 0$.

1 Introduction

In this paper we will consider cancellation in the reliability polynomial for cyclic consecutive systems. There are many different naturally occurring situations that involve the reliability of a system [2]. As one example, consider the communication system shown in Figure 1. A message is to be sent from vertex s to vertex t. Since the communication links are failure prone, we are interested in calculating the *st-reliability* of the system: the probability that at a random instant there will exist a path of operating links joining s to t. Assume that each link a_j fails randomly, and independently, with probability $q_j = 1 - p_j$; that is $p_j = \Pr[a_j]$ and $q_j = \Pr[\bar{a}_j]$.

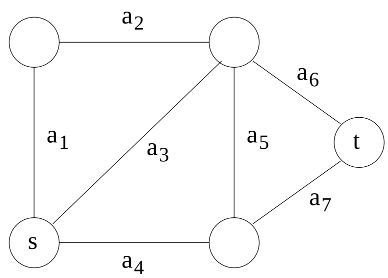

Figure 1: A communication system

In this system, there are several st-paths, namely $P_1 = \{a_3, a_6\}$, $P_2 = \{a_4, a_7\}$, $P_3 = \{a_1, a_2, a_6\}$, $P_4 = \{a_3, a_5, a_7\}$, $P_5 = \{a_4, a_5, a_6\}$, and $P_6 = \{a_1, a_2, a_5, a_7\}$. Notice that we need not consider a non-simple (non-minimal) path such as $Q = \{a_1, a_2, a_3, a_4, a_7\}$ since $P_2 \subset Q$ and the availability of path Q implies the availability of path P_2. For the system $\mathcal{S}$ defined by the collection $\{P_1,\ P_2,\ P_3,\ P_4,\ P_5,\ P_6\}$ of minimal st-paths, the availability of any such path is sufficient to ensure that the entire system $\mathcal{S}$ operates (a message can be successfully transmitted from s to t). Let E_i denote the event in which all links in path P_i are operating. By independence, $\Pr[E_i] = \prod\{p_e : a_e \in P_i\}$. The reliability $R[\mathcal{S}]$ of the system $\mathcal{S}$ can thus be expressed as

$$R[\mathcal{S}] \quad = \quad \Pr[E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6]$$

By applying the inclusion-exclusion principle [1] the reliability of the system can be evaluated using

$$R[\mathcal{S}] = \sum_i \Pr[E_i] - \sum_{i<j} \Pr[E_i E_j] + \sum_{i<j<k} \Pr[E_i E_j E_k] - \cdots - \Pr[E_1 E_2 E_3 E_4 E_5 E_6]$$

The inclusion-exclusion principle expresses the reliability of this system in terms of a polynomial in the variables p_j. Since each link operates independently of the other links, any term of this expression will be easy to calculate. However there can be up to $2^6 - 1 = 63$ terms in the expanded reliability polynomial. In this example, however, the reliability polynomial simplifies to

$$\begin{aligned} R[\mathcal{S}] \quad = \quad & p_3p_6 + p_4p_7 + p_1p_2p_6 + p_3p_5p_7 + p_4p_5p_6 + p_1p_2p_5p_7 \\ & -p_1p_2p_3p_6 - p_3p_4p_5p_6 - p_3p_4p_5p_7 - p_3p_4p_6p_7 - p_3p_5p_6p_7 - p_4p_5p_6p_7 \\ & -p_1p_2p_4p_5p_6 - p_1p_2p_3p_5p_7 - p_1p_2p_4p_5p_7 - p_1p_2p_4p_6p_7 - p_1p_2p_5p_6p_7 \\ & +p_1p_2p_3p_4p_5p_6 + p_1p_2p_3p_4p_5p_7 + p_1p_2p_3p_4p_6p_7 + p_1p_2p_3p_5p_6p_7 \\ & +2p_3p_4p_5p_6p_7 + 2p_1p_2p_4p_5p_6p_7 - 2p_1p_2p_3p_4p_5p_6p_7 \end{aligned}$$

Of importance is that only 24 of the possible 63 terms appear in the simplified reliability polynomial. In addition many of the coefficients are either ± 1. The cancellation of terms in the reliability polynomial and the ± 1 property was first studied by Satyanarayana and Prabhakar [3] for the problem of finding the *st*-reliability in a network.

Throughout this paper we will apply known techniques to better understand a specific class of systems $\mathcal{S} = \{P_1,\ P_2,\ \ldots,\ P_k\}$, defined by a collection of *minimal operating sets* P_i involving the *elements* a_j. In particular the inclusion-exclusion principle [1] will facilitate expressing the reliability of the system $\mathcal{S}$ as a polynomial:

$$R[\mathcal{S}] = \sum_i \Pr[E_i] - \sum_{i<j} \Pr[E_i E_j] + \cdots + (-1)^{k+1} \Pr[E_1 E_2 E_3 \ldots E_k] \qquad (1)$$

As assumed earlier each element operates independently of the other elements. Therefore each term of (1) will be easy to calculate. However there can be up to $2^k - 1$ terms in the resulting polynomial.

An equivalent polynomial representation of (1) can also be obtained using the $\oplus$ operator. This notational convenience will allow for a simplified way of combining multiple reliability polynomials. Therefore an alternative way of expressing the reliability of the system $\mathcal{S}$ is

$$\begin{aligned} R[\mathcal{S}] &= \Pr[E_1 \cup E_2 \cup \cdots \cup E_k] \\ &= \Pr[E_1] \oplus \Pr[E_2] \oplus \cdots \oplus \Pr[E_k] \end{aligned}$$

where in general $\Pr[E_i] \oplus \Pr[E_j]$ combines the reliability polynomials associated with the events E_i and E_j. Specifically $\Pr[E_i] \oplus \Pr[E_j] = \Pr[E_i] + \Pr[E_j] - \Pr[E_i] \otimes \Pr[E_j]$ where $\Pr[E_i] \otimes \Pr[E_j]$ constructs the polynomial term involving all the terms that appear in either $\Pr[E_i]$, $\Pr[E_j]$, or both, and any higher power term is reduced to the first power. As an example let $T_1 = \{a_1, a_2\}$ and $T_2 = \{a_2, a_3, a_4\}$, with $\mathcal{S} = \{T_1,\ T_2\}$. Then $\Pr[E_1] = p_1 p_2$, $\Pr[E_2] = p_2 p_3 p_4$, and $R[\mathcal{S}]$ is computed as follows:

$$\begin{aligned} R[\mathcal{S}] &= \Pr[E_1 \cup E_2] \\ &= \Pr[E_1] \oplus \Pr[E_2] \\ &= p_1 p_2 \oplus p_2 p_3 p_4 \\ &= p_1 p_2 + p_2 p_3 p_4 - p_1 p_2 \otimes p_2 p_3 p_4 \\ &= p_1 p_2 + p_2 p_3 p_4 - p_1 p_2 p_3 p_4 \end{aligned}$$

The focus of this paper will be to explain why cancellation occurs in the reliability polynomials for a specific class of systems. We will also address the ± 1 property for the coefficients of the polynomial terms. The goal is to expand upon some of the work of Shier and McIlwain [4] which explained the ± 1 property and some of the resulting cancellation for *linear consecutive systems*: namely, systems in which each set S_i contains elements that are consecutive integers. The class of systems considered here will by contrast be defined relative to a k-cycle. Accordingly, a system is *cyclically consecutive* if each set S_i contains elements that are consecutive integers modulo k, where remainders are to be taken in the range 1 to k. A final restriction is that all of the k sets to be considered are uniform, so each set has the same number of elements.

We will determine the coefficients for all of the polynomial terms in (1) and as well provide a method for counting the number of non-zero polynomial terms in cyclic consecutive systems. In this way, we will be able to explain all of the cancellation occurring within the reliability polynomial. Additionally by representing the structure of such systems in terms of binary strings we can investigate the corresponding generating function for the number of non-zero terms remaining after cancellation has occurred. Specifically, the asymptotic growth of the number of non-zero polynomial terms will be given by α^k, where α is the largest root to the complementary equation for the dominant term in the denominator of the generating function.

2 Analysis of Cyclic Consecutive Systems C_k^r

The structure considered in this paper is a cyclic consecutive system on elements $\{a_1, a_2, \ldots, a_k\}$, in which each of the minimal operating sets S_i has size r. Each set S_i consists of elements $\{a_i, a_{i+1}, \ldots, a_{i+r}\}$, again where the subscripts are taken modulo k. Thus we can express this system C_k^r as

$$
\begin{aligned}
S_1 &= \{a_1, a_2, \dots, a_r\} \\
S_2 &= \{a_2, a_3, \dots, a_{r+1}\} \\
&\vdots \\
S_{k-2} &= \{a_1, \dots, a_{r-3}, a_{k-2}, a_{k-1}, a_k\} \\
S_{k-1} &= \{a_1, \dots, a_{r-2}, a_{k-1}, a_k\} \\
S_k &= \{a_1, \dots, a_{r-1}, a_k\}
\end{aligned}
$$

Such a cyclic consecutive system is illustrated by the diagram in Figure 2, indicating that the sets S_i are all consecutive along the cycle. Recall that element a_j fails independently with probability $q_j = 1 - p_j$. The system $\mathcal{S} = \{S_1,\ S_2,\ \dots,\ S_k\}$ operates if there exists at least one set S_i containing all working elements.

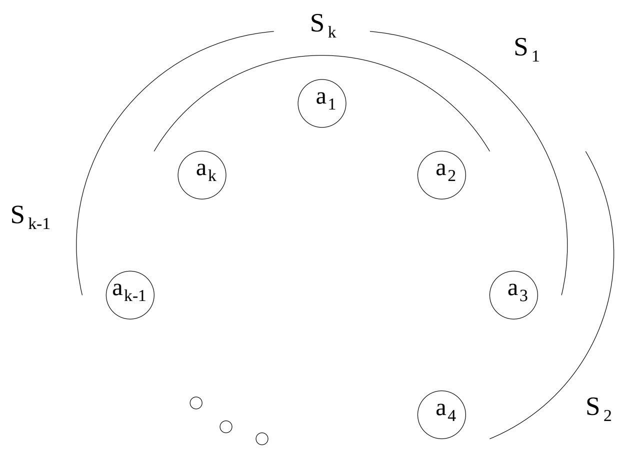

Figure 2: A cyclic consecutive system C_k^3

We will find the reliability polynomial $R[\mathcal{S}]$ for the system $\mathcal{S}$ by conditioning on the elements of any set S_i; without loss of generality we condition on the first element (if any) that fails in S_1. The reliability polynomial is then given by

$$
\begin{aligned}
R[\mathcal{S}] &= (1-p_1)R[\mathcal{S}|\bar{a}_1] + p_1(1-p_2)R[\mathcal{S}|a_1\bar{a}_2] + \cdots \\
&\quad + p_1p_2\dots p_{r-1}(1-p_r)R[\mathcal{S}|a_1a_2\dots a_{r-1}\bar{a}_r] + p_1p_2\dots p_r R[\mathcal{S}|a_1a_2\dots a_r]
\end{aligned} \tag{2}
$$

Here $\mathcal{S}|\bar{a}_1$ corresponds to the system $\mathcal{S}$ with a_1 failing to operate. Similarly $\mathcal{S}|a_1\bar{a}_2$ corresponds to the system where a_1 is known to operate but a_2 fails, and so forth. Relative to the reliability polynomial $R[\mathcal{S}]$ the first objective is to find the coefficient of its *maximal term*, namely the coefficient of $p_1p_2\ldots p_k$ in $R[\mathcal{S}]$. This coefficient is denoted $[p_1p_2\ldots p_k]R[\mathcal{S}]$. By equating coefficients of (2) we obtain

$$[p_1\ldots p_k]R[\mathcal{S}] = -[p_2\ldots p_k]R[\mathcal{S}_1] - [p_3\ldots p_k]R[\mathcal{S}_2] - \cdots - [p_{r+1}\ldots p_k]R[\mathcal{S}_r] \tag{3}$$

where $\mathcal{S}_1$ is the system $\mathcal{S}|\bar{a}_1$, $\mathcal{S}_2$ is the system $\mathcal{S}|a_1\bar{a}_2$, and in general $\mathcal{S}_i$ is the system $\mathcal{S}|a_1\ldots a_{i-1}\bar{a}_i$. The polynomial term $p_1p_2\ldots p_rR[\mathcal{S}|a_1a_2\ldots a_r]$ simplifies to $p_1p_2\ldots p_r$ since $R[\mathcal{S}|a_1a_2\ldots a_r] = 1$; it therefore will not have a contribution when determining the coefficient of $p_1p_2\ldots p_k$ in $R[\mathcal{S}]$. Of significance is that each of these systems $\mathcal{S}_i$ are *linear* consecutive systems. This will be seen presently by looking at the structure of the systems with respect to the success or failure of different elements.

System $\mathcal{S}_1$ is defined by the collection of minimal operating sets of $\mathcal{S}$ that remain when a_1 fails. Since the failure of a_1 eliminates the sets $S_1,\ S_{k-r+2},\ S_{k-r+3},\ \ldots,\ S_k$, it follows that $\mathcal{S}_1 = \{\mathcal{S}_2,\ \mathcal{S}_3,\ \ldots,\ \mathcal{S}_{k-r+1}\}$. Thus $\mathcal{S}_1$ is a linear consecutive system, involving elements $a_2,\ \ldots,\ a_k$ and containing $k-r$ minimal operating sets, each of size r; such a system is denoted L^r_{k-r}. To obtain $[p_2\ldots p_k]R[\mathcal{S}_1]$ in (3) we require the coefficient $\mu(L^r_{k-r})$ of the maximal term in the reliability polynomial for L^r_{k-r}, giving

$$[p_2\ldots p_k]R[\mathcal{S}_1] = \mu(L^r_{k-r})$$

In general $\mu(\mathcal{S})$ will denote the coefficient of the maximal term for a given system $\mathcal{S}$.

System $\mathcal{S}_2$ is the collection of minimal operating sets that remain when a_1 works and a_2 fails. The failure of a_2 eliminates the operating sets $S_1,\ S_2,\ S_{k-r+3},\ \ldots,\ S_k$. With a_1 operating the set S_{k-r+2} is reduced to $S_{k-r+2} = \{a_{k-r+2},\ldots,a_k\}$. Consequently the set $S_{k-r+1} = \{a_{k-r+1},\ldots,a_k\}$ is no longer minimal and is *absorbed* by the set $S_{k-r+2} \subset S_{k-r+1}$. This means that the system S_2 can be expressed as $\mathcal{S}_2 = \{S_3,\ \ldots,\ S_{k-r},\ S_{k-r+2}\}$, a linear consecutive system involving elements $a_3,\ \ldots,\ a_k$.

We will find the coefficient $[p_3\ldots p_k]R[\mathcal{S}_2]$ by generating the maximal term $p_3\ldots p_k$ in $R[\mathcal{S}_2]$. To do so we decompose $\mathcal{S}_2$ into the two linear consecutive subsystems $\mathcal{A}_1 = \{S_3,\ \ldots,\ S_{k-r}\}$ and $\mathcal{A}_2 = \{S_{k-r+2}\}$. When the reliability polynomials for subsystems $\mathcal{A}_1$ and $\mathcal{A}_2$ are combined the resulting reliability polynomial will have as its maximal term the term of interest, namely

$p_3 p_4 \ldots p_k$. Combining these two reliability polynomials produces

$$\begin{aligned} R[\mathcal{S}_2] &= R[\mathcal{A}_1] \oplus R[\mathcal{A}_2] \\ &= R[\mathcal{A}_1] + R[\mathcal{A}_2] - R[\mathcal{A}_1] \otimes R[\mathcal{A}_2] \end{aligned}$$

By equating the coefficients of $p_3 p_4 \ldots p_k$ we have

$$\mu(\mathcal{S}_2) = 0 + 0 - \mu(R[\mathcal{A}_1] \otimes R[\mathcal{A}_2])$$

A first observation is that there are $r-2$ elements in the overlap of the subsystems $\mathcal{A}_1$ and $\mathcal{A}_2$; namely $\mathcal{A}_1 \cap \mathcal{A}_2 = \{a_{k-r+2}, \ldots, a_{k-1}\}$. This will allow $r-1$ different ways to obtain the maximal term $p_3 \ldots p_k$ in $R[\mathcal{S}_2]$. These $r-1$ ways are given by combining any one of the terms $[p_3 \ldots p_{k-1}]R[\mathcal{A}_1]\ p_3 \ldots p_{k-1},\ [p_3 \ldots p_{k-2}]R[\mathcal{A}_1]\ p_3 \ldots p_{k-2},\ \ldots,\ [p_3 \ldots p_{k-r+1}]R[\mathcal{A}_1]\ p_3 \ldots p_{k-r+1}$ from $R[\mathcal{A}_1]$ with the term $p_{k-r+2} \ldots p_k = R[\mathcal{A}_2]$. Since $[p_3 \ldots p_{k-1}]R[\mathcal{A}_1] = \mu(L^r_{k-(r+2)})$, the corresponding coefficient for $p_3 \ldots p_k$ in $R[\mathcal{S}_2]$ is $-\mu(L^r_{k-(r+2)})$. Likewise the coefficient for $p_3 \ldots p_{k-2}$ is $[p_3 \ldots p_{k-2}]R[\mathcal{A}_1] = \mu(L^r_{k-(r+3)})$, producing the coefficient $-\mu(L^r_{k-(r+3)})$ for $p_3 \ldots p_k$ in $R[\mathcal{S}_2]$. Similarly, the coefficient for $p_3 \ldots p_{k-r+1}$ in $R[\mathcal{A}_1]$ is $\mu(L^r_{k-2r})$ which produces the coefficient $-\mu(L^r_{k-2r})$ for $p_3 \ldots p_k$ in $R[\mathcal{S}_2]$. Consequently $\mu(\mathcal{S}_2)$ is found as $-\mu(L^r_{k-(r+2)}) - \mu(L^r_{k-(r+3)}) - \cdots - \mu(L^r_{k-2r})$, giving a contribution in (3) of

$$-[p_3 \ldots p_k]R[\mathcal{S}_2] = \mu(L^r_{k-(r+2)}) + \mu(L^r_{k-(r+3)}) + \cdots + \mu(L^r_{k-2r})$$

In general for $i \geq 2$ the term $-[p_{i+1} \ldots p_k]R[\mathcal{S}_i]$ will contribute $\mu(L^r_{k-2r}) + \mu(L^r_{k-2r+1}) + \cdots + \mu(L^r_{k-(r+i)})$ to the coefficient of the maximal term in (3). This can be seen by looking at the structure of $\mathcal{S}_i = \mathcal{S}|a_1 \ldots a_{i-1}\bar{a}_i$. The failure of a_i eliminates the r sets $S_1,\ \ldots,\ S_i,\ S_{k-r+i+1},\ \ldots,\ S_k$. Also since $a_1, \ldots, a_{i-1}$ are operating, certain of the sets are no longer minimal. Specifically the sets $S_{k-r+1},\ \ldots,\ S_{k-r+i-1}$ are no longer minimal, and are absorbed by the set $S_{k-r+i} = \{a_{k-r+i}, \ldots, a_k\}$, giving $\mathcal{S}_i = \{S_{i+1},\ \ldots,\ S_{k-r},\ S_{k-r+i}\}$. To find $[p_{i+1} \ldots p_k]R[\mathcal{S}_i]$ we again decompose $\mathcal{S}_i$ into two subsystems $\mathcal{A}_1 = \{S_{i+1},\ \ldots,\ S_{k-r}\}$ and $\mathcal{A}_2 = \{S_{k-r+i}\}$. Note that $\mathcal{A}_1 \cap \mathcal{A}_2$ have in common the $r-i$ elements $a_{k-r+i}, \ldots, a_{k-1}$. If as before we combine the two reliability polynomials $R[\mathcal{A}_1]$ and $R[\mathcal{A}_2]$, then the resulting reliability polynomial $R[\mathcal{A}_1] \oplus R[\mathcal{A}_2]$ will have $p_{i+1} \ldots p_k$ as its maximal term. Equating coefficients of $p_{i+1} \ldots p_k$ again produces $\mu(\mathcal{S}_i) = -\mu(R[\mathcal{A}_1] \otimes R[\mathcal{A}_2])$. The $r-i$ elements in the overlap of $\mathcal{A}_1$ and $\mathcal{A}_2$ determine $r-i+1$ ways to obtain $p_{i+1} \dots p_k$ in $R[\mathcal{S}_i]$. Namely, each of the $r-i+1$ terms $p_{i+1} \dots p_{k-(r-i+1)}$, $p_{i+1} \dots p_{k-(r-i)}$, $\dots$, $p_{i+1} \dots p_{k-1}$ from $R[\mathcal{A}_1]$ can be combined with the term $p_{k-r+i} \dots p_k$ from $R[\mathcal{A}_2]$. These terms from $R[\mathcal{A}_1]$ correspond, respectively, to maximal terms from the linear consecutive systems L^r_{k-2r}, L^r_{k-2r+1}, $\dots$, L^r_{k-r-i} and so $\mu(\mathcal{S}_i) = -\left[\mu(L^r_{k-2r}) + \mu(L^r_{k-2r+1}) + \cdots + \mu(L^r_{k-(r+i)})\right]$. Combining these contributions for all systems $\mathcal{S}_i$, i from 1 to r, we obtain the coefficient of the maximal term $[p_1 \dots p_k]R[\mathcal{S}]$ in (3) as

$$\mu(C^r_k) = -\mu(L^r_{k-r}) + \sum_{j=2}^{r} (j-1)\mu(L^r_{k-(r+j)}) \tag{4}$$

Therefore, one can easily generate $[p_1 \dots p_k]R[\mathcal{S}]$ once we establish the maximal coefficients $\mu(L^r_w)$ for linear consecutive systems. We now find an explicit formula for these coefficients.

Recall that L^r_w is the system composed of w linear consecutive sets T_i each of size r. It is convenient to order the sets T_i oppositely from the situation when working with cyclic consecutive sets. Consequently L^r_w will be represented as

$$\begin{aligned}
T_w &= \{b_1, b_2, \dots, b_r\} \\
T_{w-1} &= \{b_2, b_3, \dots, b_{r+1}\} \\
&\vdots \\
T_1 &= \{b_w, b_{w+1}, b_{w+r-1}\}
\end{aligned}$$

To find the coefficient of the maximal term in L^r_w we look at the associated undirected graph $T(L^r_w)$ defined in Shier and McIlwain [4]. Namely $T(L^r_w)$ is an undirected tree on $w+1$ nodes with node $i \neq w+1$ corresponding to the set T_i and node $w+1$ added as the root node. For each i, the tree edge (i,j), with $i < j$, represents the situation when $T_i \cup T_{j-1}$ is consecutive but $T_i \cup T_j$ is not consecutive. Notationally this will occur when $j = m_i + 1$ and m_i is the largest index for which $T_i \cup T_{m_i}$ is consecutive. Specifically, we obtain the following values for the system L^r_w:

$$\begin{aligned}
m_i &= i + r \quad \text{for } i = 1, \ \dots, \ w-(r+1) \\
m_i &= w \qquad \text{for } i = w-r, \ \dots, \ w
\end{aligned}$$

Thus in the tree $T(L^r_w)$ node i will be connected to node $j = i+r+1$ for $i = 1, \ \dots, \ w-(r+1)$. Also nodes $w, \ w-1, \ \dots, \ w-r$ will each be connected to node $w+1$.

The graph so constructed is a star-like tree rooted at node $w+1$. For convenience we label the r subtrees of this tree $A_1, A_2, \ldots, A_r$. Figure 3 gives such a tree for the case $r=3$. Notice that all of the nodes in a subtree are congruent mod $r+1$. The subtrees are labeled so that A_i contains node $w+1-i$. To find the coefficient $[b_1 \ldots b_{w+r-1}]R[L_w^r]$ we examine the unique path P_{12} from node 1 to node 2 in the constructed tree $T(L_w^r)$. Specifically we will be interested in whether this path contains the edge $(w, w+1)$. If so, Shier and McIlwain [4] have shown that $[b_1 \ldots b_{w+r-1}]R[L_w^r]$ is given by $(-1)^{|P_{12}|+1}$, where $|P_{12}|$ is the number of edges in P_{12}; otherwise this coefficient is zero. A first observation is that any path from node 1 to node 2 must go through node $w+1$, since nodes 1 and 2 are in different subtrees A_i. Since A_1 contains node w, edge $(w, w+1)$ is in P_{12} if and only if either node 1 or node 2 is in the set A_1. This happens precisely when $w \equiv 1 \pmod{r+1}$ or $w \equiv 2 \pmod{r+1}$.

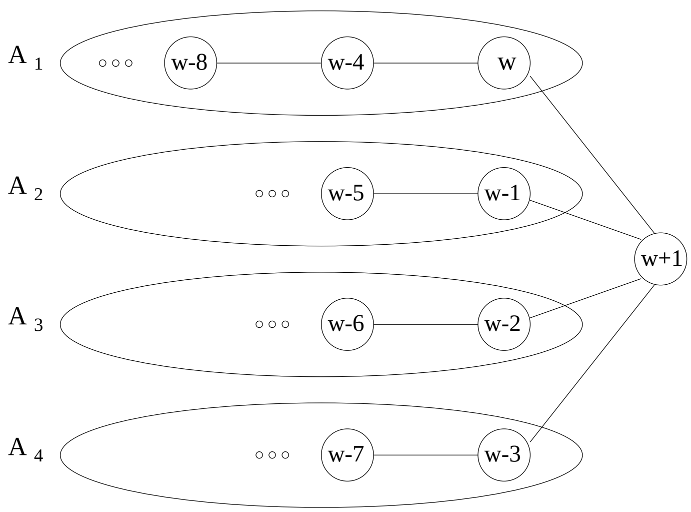

Figure 3: The rooted tree $T(L_w^3)$

¿From the way that the tree $T(L_w^r)$ is constructed we can easily calculate the length of the path $P_{i,w+1}$ from node i to node $w+1$, for $i = 1, \ldots, r+1$:

$$|P_{i,w+1}| = \left\lfloor \frac{w-i}{r+1} \right\rfloor + 1 \tag{5}$$

Therefore when $w \equiv 1 \pmod{r+1}$ then $|P_{12}|$ will be odd since $|P_{12}| = |P_{1,w+1}| + |P_{2,w+1}| = \lfloor \frac{w-1}{r+1} \rfloor + 1 + \lfloor \frac{w-2}{r+1} \rfloor + 1$ and $\lfloor \frac{w-1}{r+1} \rfloor = \lfloor \frac{w-2}{r+1} \rfloor + 1$. In this case $[b_1 \ldots b_{w+r-1}]R[L_w^r] = (-1)^{|P_{12}|+1} = 1$. When $w \equiv 2 \pmod{r+1}$ then $|P_{12}|$ will be even since $|P_{12}| = \lfloor \frac{w-1}{r+1} \rfloor + 1 + \lfloor \frac{w-2}{r+1} \rfloor + 1$ and $\lfloor \frac{w-1}{r+1} \rfloor = \lfloor \frac{w-2}{r+1} \rfloor$. Therefore $[b_1 \ldots b_{w+r-1}]R[L_w^r] = (-1)^{|P_{12}|+1} = -1$. When $w \not\equiv 1 \pmod{r+1}$

and $w \not\equiv 2 \pmod{r+1}$ we have that $[b_1 \ldots b_{w+r-1}]R[L^r_w] = 0$ since edge $(w, w+1)$ is not in P_{12}. In summary, the coefficient of the maximal term in L^r_w is given by

$$\mu(L^r_w) = \begin{cases} +1 & \text{if } w \equiv 1 \pmod{r+1} \\ -1 & \text{if } w \equiv 2 \pmod{r+1} \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Having established all the values for $\mu(L^r_w)$ we can return to the original problem of finding $[p_1 \ldots p_k]R[\mathcal{S}]$. Since the coefficients of the maximal terms in L^r_w are determined by the value of $w \pmod{r+1}$, similar patterns will occur for $[p_1 \ldots p_k]R[\mathcal{S}]$ when looking now at $k \pmod{r+1}$, where $k = |\mathcal{S}|$ refers to the number of sets in the system $\mathcal{S}$. For the system C^r_k, substitution of the values from (6) into (4) and simplification give the following result, which we state as a theorem.

Theorem 1 *Let $\mathcal{S} = \{S_1,\ S_2,\ \ldots,\ S_k\}$ be the system C^r_k. Then*

$$\mu(C^r_k) = \begin{cases} -r & k \equiv 0 \ (mod\ r+1) \\ 1 & otherwise \end{cases} \tag{7}$$

For illustration we examine the case in (7) when $k \equiv 0 \pmod{r+1}$. The first term $-\mu(L^r_{k-r})$ in equation (4) will have $k-r \equiv 1 \pmod{r+1}$ which results in $-\mu(L^r_{k-r}) = -1$. The only other term in equation (4) that will have a non-zero coefficient occurs when $j = r$. None of the other values of $j = 2, \ldots,\ r-1$ result in $k - (r+j) \equiv 1$ or $2 \pmod{r+1}$. When $j = r$ then $\mu(L^r_{k-2r}) = -1$ since $k - 2r \equiv 2 \pmod{r+1}$. Therefore $\mu(C^r_k) = -1 + (r-1)(-1) = -r$ as stated in (7).

Equation (6) will also facilitate the analysis of *non-maximal* consecutive terms in $R[\mathcal{S}]$. The coefficient for a consecutive term of length n, where $r \leq n < k$, is found as $\mu(L^r_{n-r+1})$, since each term of length n can be generated as the maximal term of a linear consecutive system with $n - (r-1) = n - r + 1$ sets. Only those terms where $\mu(L^r_{n-r+1}) \neq 0$ will yield contributions to $R[\mathcal{S}]$. A straightforward computation, which involves counting the number of relevant non-zero coefficients of $\mu(L^r_w)$, reveals that when $k \equiv 0 \pmod{r+1}$ there are $2\frac{k}{r+1} - 1$ different lengths of non-maximal consecutive terms appearing with a non-zero coefficient. For all other values of k there are $2\lfloor\frac{k}{r+1}\rfloor$ different lengths of non-maximal consecutive terms with a non-zero coefficient. Notice that there are exactly k consecutive terms for a given length, each corresponding to a different starting point on the cycle. In addition the maximal term always appears in the reliability polynomial. Thus the total number (CTERMS) of consecutive terms is given by the following result which we state as a theorem.

Theorem 2 *The total number of consecutive non-zero polynomial terms in the system* $\mathcal{S} = C_k^r$ *is*

$$CTERMS = \begin{cases} k(2\frac{k}{r+1} - 1) + 1 & k \equiv 0 \ (mod\ r+1) \\ k(2\lfloor \frac{k}{r+1} \rfloor) + 1 & otherwise \end{cases} \tag{8}$$

Any *non-consecutive* term ψ will contain $\alpha \geq 2$ disjoint maximal consecutive terms. Relative to the non-consecutive term ψ we need only consider those sets S_i that have all of their elements contained in ψ. These sets naturally form $\alpha \geq 2$ linear consecutive subsystems $\mathcal{S}_i$, $i = 1$ to α, corresponding to each disjoint maximal consecutive term. Equation (6) gives us the coefficient $\mu(L_{w_i}^r)$ for each maximal consecutive term of length n_i, where $w_i = n_i - r + 1$. To construct the term ψ we now apply the inclusion-exclusion principle to these linear consecutive subsystems $\mathcal{S}_i$; the resulting polynomial will have ψ as its maximal term. It is important to note that the only way to construct ψ is to incorporate the maximal term from each linear consecutive subsystem. Therefore by using the inclusion-exclusion principle the coefficient for ψ can be expressed in terms of the maximal coefficient $\mu(L_{w_i}^r)$ for each disjoint maximal consecutive term, with the sign inherited from equation (1). Namely the coefficient of the non-consecutive term ψ can be expressed as

$$[\psi]R[\mathcal{S}] = (-1)^{\alpha+1} \prod_{i=1}^{\alpha} \mu(L_{w_i}^r) \tag{9}$$

Clearly each of these coefficients will be ± 1 or 0 as the maximal coefficient for a linear consecutive system in ± 1 or 0.

The next task is to count the number of non-consecutive terms ψ that appear with a non-zero coefficient in the reliability polynomial so that we can fully determine the amount of cancellation that occurs in $R[\mathcal{S}]$. To do so we need to look at which non-consecutive terms can appear in $R[\mathcal{S}]$. First, in order for a non-consecutive term to appear each of its constituent maximal consecutive terms must have $\mu(L_{w_i}^r) \neq 0$, where the consecutive term has length $n_i = w_i + r - 1$. In addition, there are restrictions on the total length $n = \sum n_i$ of the term. Since ψ has been decomposed into α disjoint (maximal) consecutive terms of length n_i, then $\sum_{i=1}^{\alpha}(n_i + 1) \leq k$ holds as there must be a "gap" between each constituent term on the cycle. Equivalently, we define the *block size* to have length $l_i = n_i + 1 = w_i + r$ and only consider blocks for which $\mu(L_{n_i - r+1}^r) \neq 0$, as determined by (6): namely $n_i \equiv -1, 0 \pmod{r+1}$. Suppose that these possible block sizes are given by $l_1,\ l_2,\ \ldots$, listed in order of increasing size l_i. To determine which combinations of blocks result in a feasible non-consecutive term we need to find all combinations $\Psi_j = (a_{1j},\ a_{2j},\ \ldots)$ where $a_{ij} \in \mathbb{Z}^+$ and

$$\sum_{i\geq 1} a_{ij} l_i \leq k, \quad \sum_{i\geq 1} a_{ij} \geq 2 \tag{10}$$

Here a block of size l_i occurs a_{ij} times in the non-consecutive term and $\sum\limits_i a_{ij} = \alpha_j$, the number of disjoint maximal consecutive terms for the subpartition Ψ_j of k. Let $|\Psi_j|$ denote the number of non-consecutive terms having the specified subpartition structure Ψ_j. For a given Ψ_j recall that $n = \sum n_i$ is the length of an associated non-consecutive term. Once we place the first maximal consecutive term of Ψ_j on the cycle we need to make sure that there is a gap between each of the maximal consecutive terms. There are $k-n$ gap elements to be placed around the cycle and we need α_j gaps separating the α_j maximal consecutive terms. Equivalently, we want to place $k-n$ balls, the gap elements, into α_j urns, the gaps, with each urn being non-empty; this can be done in precisely $\binom{k-n-1}{\alpha_j-1}$ ways. However we have not taken into account all of the different arrangements of the $\alpha_j - 1$ maximal consecutive terms other than the first. Consequently we need to multiply the above count by $(\alpha_j - 1)!$. Without worrying about where the first maximal consecutive term begins on the k-cycle we have $\binom{k-n-1}{\alpha_j-1}(\alpha_j-1)! = (k-n-1)_{\alpha_j-1}$ placements of the remaining $\alpha_j - 1$ maximal consecutive terms, where $(x)_m$ denotes a falling factorial. Initially we arbitrarily placed the first maximal consecutive term on the k-cycle. There are k choices for where its first element could be placed. However, double counting occurs whenever there are maximal consecutive terms of the same length. To eliminate this overcounting we divide by the factorial of the number of maximal consecutive terms with the same length, producing

$$|\Psi_j| = \frac{(k-n-1)_{\alpha_j-1}}{\prod\limits_{i\geq 1} a_{ij}!}\, k \tag{11}$$

The total number (NTERMS) of non-consecutive terms is found by summing (11) over all j, that is over all subpartitions Ψ_j of k, which yields the following theorem.

Theorem 3 *The total number of non-consecutive non-zero polynomial terms in the system $\mathcal{S} = C_k^r$ is*

$$NTERMS = \sum_j \left(\frac{(k-n-1)_{\alpha_j-1}}{\prod\limits_{i\geq 1} a_{ij}!}\, k \right) \tag{12}$$

The formula (12) will be illustrated by computing NTERMS for C_{10}^3. We begin by observing which linear consecutive systems L_w^3 have a non-zero maximal coefficient and the corresponding length of the maximal term. Table 1 shows the relevant linear systems and their block sizes. The next step is to find all the subpartitions Ψ_j of $k = 10$ resulting in a non-consecutive term. The three possibilities that satisfy the constraints (10) are $\Psi_1 = (2,0,0,0)$, $\Psi_2 = (1,1,0,0)$, and $\Psi_3 = (0,2,0,0)$. Here $\alpha_1 = \alpha_2 = \alpha_3 = 2$. For Ψ_1 the length of the non-consecutive term is $n = 6$ resulting in $|\Psi_1| = \frac{(10-6-1)}{2!0!0!0!}10 = 15$. Likewise for Ψ_2 we have $n = 7$ resulting in $|\Psi_2| = \frac{(10-7-1)}{1!1!0!0!}10 = 20$ and for Ψ_3 we have $n = 8$ producing $|\Psi_3| = \frac{(10-8-1)}{0!2!0!0!}10 = 5$. Therefore, in C_{10}^3 we have NTERMS $= \sum_j |\Psi_j| = 15 + 20 + 5 = 40$.

Table 1: Development of NTERMS for $R[C_{10}^3]$

w	1	2	5	6
$\mu(L_w^3)$	1	-1	1	-1
n	3	4	7	8
block size	l_1=4	l_2=5	l_3=8	l_4=9

Equations (8) and (12) enable counting the total number of non-zero terms in the simplified reliability polynomial $R[\mathcal{S}]$. Table 2 shows the results obtained for several cyclic consecutive systems; it clearly demonstrates the high degree of cancellation that occurs compared to the potential number in the inclusion-exclusion expansion (1). For example in the system C_{15}^2 only $\frac{2744}{32767} \approx 8.37\%$ of the possible terms remain after cancellation and for the system C_{15}^6 only $\frac{91}{32767} \approx 0.28\%$ of the possible terms appear. In addition all of the non-maximal non-zero terms have coefficients corresponding to the maximal coefficient of a linear consecutive system, or the product of such coefficients. As these maximal coefficients of linear consecutive systems are ± 1 all of the terms appearing in $R[C_k^r]$, except for possibly the maximal term, must also have a coefficient of ± 1.

3 Generating Functions

We now present an alternative approach to facilitate counting the number of non-zero polynomial terms in the system C_k^r using binary strings of length k and their corresponding generating

Table 2: Number of terms in $R[C_k^r]$				
C_k^r	CTERMS	NTERMS	Total	Possible (2^k-1)
C_8^2	33	32	65	255
C_8^3	25	4	29	255
C_8^4	17	0	17	255
C_8^5	17	0	17	255
C_8^6	17	0	17	255
C_{10}^2	61	140	201	1023
C_{10}^3	41	40	81	1023
C_{10}^4	31	5	36	1023
C_{10}^5	21	0	21	1023
C_{10}^6	21	0	21	1023
C_{15}^2	136	2608	2744	32767
C_{15}^3	91	580	671	32767
C_{15}^4	76	170	246	32767
C_{15}^5	61	90	151	32767
C_{15}^6	61	30	91	32767

functions. If we allow a 1 to correspond to the event that an element operates, and a 0 otherwise, each polynomial term then corresponds to blocks of 1's and 0's. By restricting our binary strings to those beginning with a 1 and ending with a 0 we can represent all such possible binary strings by the following representation: $((1^r \cup 1^{r+1})(1^{r+1})^*00^*)((1^r \cup 1^{r+1})(1^{r+1})^*00^*)^*$. Our first block of 1's, and also any other subsequent block of 1's, must have length congruent to -1 or $0 \pmod{r+1}$, as these are the only possible lengths for non-zero linear consecutive terms. The 0's then correspond to gaps between linear consecutive terms. The preceding binary representation has the following generating function: $g(x) = \frac{1}{1-z(x)}$, where $z(x) = (x^r + x^{r+1}) \cdot \frac{1}{1-x^{r+1}} \cdot \frac{x}{1-x}$. However this will only generate all the non-zero polynomial terms that start with p_1 and end prior to p_k. In order to differentiate between all of the k different possible starting points for such polynomial terms we need to differentiate the generating function and multiply by x. Therefore it follows that the total generating function gf is given by $gf = x \cdot \frac{d}{dx} g(x)$. Taking the derivative, multiplying by x

and simplifying results in the following expression:

$$gf = \frac{x((r+1)x^r + 2x^{r+1} - (r+1)x^{r+2} - (r+1)x^{2r+1} - 4x^{2r+2} + (r+1)x^{2r+3} + 2x^{3r+3})}{(1-x^{r+1})^2(1-x-2x^{r+1})(1-x)} \tag{13}$$

Note that this will generate all possible non-zero polynomial terms except for the maximal term as there is always at least one 0 in every binary string, which corresponds to at least one element missing in any polynomial term. Substituting in specific values of r we can then extract the coefficients of x^k in the resulting series expansion. These coefficients then count the total number of non-zero polynomial terms minus 1, which corresponds to the entries in the second to last column of Table 2 minus 1.

More importantly this will allow us to find the asymptotic growth rate for these polynomial terms. The denominator of the above generating function is $(1-x^{r+1})^2(1-x-2x^{r+1})(1-x)$. The terms $(1-x^{r+1})^2 \cdot (1-x)$ contribute only $O(k^2)$, so for analyzing the asymptotic behavior we can ignore them. Thus we want the smallest root of $1-x-2x^{r+1}=0$. By standard techniques it can be shown that this polynomial has a unique root in $(0,1)$. To find this root we proceed by Lagrange Inversion [5] where we consider

$$\begin{aligned}
t^{r+1} - t^{r+1}x - x^{r+1} &= 0 \quad \text{at } t = \left(\frac{1}{2}\right)^{\frac{1}{r+1}} \\
t^{r+1} &= \frac{x^{r+1}}{1-x} \\
t &= \frac{x}{(1-x)^{\frac{1}{r+1}}} \\
x &= t(1-x)^{\frac{1}{r+1}}
\end{aligned}$$

which is now in the form $x = t\phi(x)$. Therefore the coefficients of the series are given by

$$\begin{aligned}
c_k = [t^k]x(t) &= \frac{1}{k}[\lambda^{k-1}]\phi(\lambda)^k \\
&= \frac{1}{k}[\lambda^{k-1}](1-\lambda)^{\frac{k}{r+1}} \\
&= \frac{1}{k}\binom{(\frac{k}{r+1})}{k-1}(-1)^{k-1}
\end{aligned}$$

So $x(t) = \sum_k c_k t^k$ results in

$$x\left(\left(\frac{1}{2}\right)^{\frac{1}{r+1}}\right) = \sum_{k \geq 1} \frac{1}{k}\binom{(\frac{k}{r+1})}{k-1}(-1)^{k-1}\left(\frac{1}{2}\right)^{\frac{k}{r+1}}$$

thus establishing

$$\beta = \sum_{k\geq 1} \frac{1}{k} \binom{(\frac{k}{r+1})}{k-1} (-1)^{k-1} \left(\frac{1}{2}\right)^{\frac{k}{r+1}} \tag{14}$$

is the smallest root of $1 - x - 2x^{r+1} = 0$. Before applying this root we first prove that this series does indeed converge. Clearly the term in question is $\binom{(\frac{k}{r+1})}{k-1}$ as all other terms are ≤ 1 in absolute value.

$$\begin{aligned}
\left|\binom{(\frac{k}{r+1})}{k-1}\right| &= \left|\frac{(\frac{k}{r+1})(\frac{k}{r+1}-1)\dots(\frac{k}{r+1}-n+2)}{(k-1)!}\right| \\
&< \frac{\lceil\frac{k}{r+1}\rceil!(k-1-\lfloor\frac{k}{r+1}\rfloor)!}{(n-1)!} \\
&= \frac{1}{\frac{(k-1)!}{\lceil\frac{k}{r+1}\rceil!(k-1-\lceil\frac{k}{r+1}\rceil)!}} \\
&= \frac{1}{\binom{k-1}{\lceil\frac{k}{r+1}\rceil}} \\
&< 1
\end{aligned}$$

Thus the series is dominated by a geometric series and as a result is convergent.

Given the smallest root β of the denominator we now seek to find the asymptotic growth rate for the number of non-zero polynomial terms. Then $\alpha = \frac{1}{\beta}$ will be the largest root of the complementary equation $2 + x^r - x^{r+1} = 0$. Let $g(x) = x((r+1)x^r + 2x^{r+1} - (r+1)x^{r+2} - (r+1)x^{2r+1} - 4x^{2r+2} + (r+1)x^{2r+3} + 2x^{3r+3})$, the numerator of the generating function, and $f(x) = (1-x^{r+1})^2(1-x-2x^{r+1})(1-x)$, the denominator of the generating function in (13). Additionally we know that $f(x) = (1-\alpha x)h(x)$. Then by partial fractions we have that $\frac{g(x)}{h(x)(1-\alpha x)} = \frac{c}{1-\alpha x} +$ other terms, where c will be the coefficient for the growth rate. By evaluating x near the root β we see that $g(x) \simeq g(\beta)$ and $f(x) \simeq (1-\alpha x)h(\beta)$, establishing that c can be found as $\frac{g(\beta)}{h(\beta)}$. To compute $h(\beta)$ observe that $f(x) = (1-\alpha x)h(x)$ and so differentiating $f'(x) = -\alpha h(x) + (1-\alpha x)h'(x)$. Evaluating at $x = \beta$ we have $f'(\beta) = -\alpha h(\beta)$. Therefore $h(\beta) = -\beta f'(\beta)$ and $g(\beta)$ can be computed as is. So $[x^k]\frac{g(x)}{f(x)} \sim c\alpha^k$, where $c = \frac{g(\beta)}{-\beta f'(\beta)}$. However, by observing that $g(x) + xf'(x)$ is divisible by $1 - x - 2x^{r+1}$ we have that $g(\beta)$ and $-\beta f'(\beta)$ are congruent $\mathrm{mod}(1 - x - 2x^{r+1})$. Consequently the growth rate is given simply by α^k, which we will state as a theorem.

Theorem 4 *The total number of non-zero polynomial terms in the system $\mathcal{S} = C_k^r$ is*

$$|R[C_k^r]| \sim \alpha^k \tag{15}$$

where α *is the largest root of* $2 + x^r - x^{r+1}$*, which is given by* $\frac{1}{\alpha} = \sum_{k\geq 1} \frac{1}{k}\binom{(\frac{k}{r+1})}{k-1}(-1)^{k-1}\left(\frac{1}{2}\right)^{\frac{k}{r+1}}$

Moreover, as previously stated the terms $(1-x^{r+1})^2 \cdot (1-x)$ contribute $O(k^2)$ to the asymptotic behavior. Therefore let $\alpha_0, \alpha_1, \cdots, \alpha_r$ be the roots of the complementary equation $2+x^r-x^{r+1} = 0$. Then

$$|R[C_k^r]| = \sum_{i=0}^{r} \alpha_i^k + O(k^2) \tag{16}$$

Alternatively, utilizing the fact that the coefficient for any non-zero, non-maximal, polynomial term is either $+1$ or -1 we can modify the above generating function to reflect this ± 1 property and consequently obtain the coefficient of the maximal term. Using the same binary representation $((1^r \cup 1^{r+1})(1^{r+1})^*00^*)((1^r \cup 1^{r+1})(1^{r+1})^*00^*)^*$ and incorporating the fact that any linear consecutive term with length congruent to $-1 \pmod{r+1}$ has a $+1$ coefficient and any linear consecutive term with length congruent to $0 \pmod{r+1}$ has a -1 coefficient, we obtain the generating function $g(x) = \frac{1}{1-w(x)}$, where $w(x) = (x^r - x^{r+1}) \cdot \frac{1}{1-x^{r+1}} \cdot \frac{x}{1-x} = \frac{x^{r+1}}{1-x^{r+1}}$. Again differentiating and multiplying by x results in the total generating function gf given by

$$gf = \frac{(r+1)x^{r+1}}{(1-x^{r+1})^2} \tag{17}$$

In this case all of the coefficients of x^k are either $r+1$, when $k \equiv 0 \pmod{r+1}$, or 0 otherwise. Given that the reliability for the event where each element operates must be 1 the coefficient of the maximal polynomial term is then either $-r$, when $k \equiv 0 \pmod{r+1}$, or 1 otherwise, thus giving an alternative proof of Theorem 1.

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