ON A CLASS OF APERIODIC SUM-FREE SETS

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ABSTRACT. We show that certain natural aperiodic sum-free sets are incomplete, that is that there are infinitely many n not in S which are not a sum of two elements of S.

1. Introduction

An old question (ascribed to Dickson by Guy [1] [6] and rediscovered in a more general setting by Cameron [4]) asks whether a sum-free set constructed in a greedy fashion from a finite initial set will be ultimately periodic. Cameron, Calkin and Finch have investigated this question, and it seems that the answer may be "no": in particular, the sum-free sets $\{1, 3, 8, 20, 26 \dots\}$ and $\{2, 15, 16, 23, 27 \dots\}$ are not known to be ultimately periodic. The approach taken by these authors is to compute a large number of elements of the respective sets, and to investigate the structure. Unfortunately there is currently no way to show that these sets are aperiodic (though there is good evidence [2, 3] to suggest that they are).

An alternative approach to answering Dickson's question in the negative would be to take a sum-free set already known to be aperiodic, and to show that it was complete. In this paper we shall show that this approach will fail for a natural set of aperiodic sum-free sets.

2. Definitions

Given a set S of positive integers, we denote by $S + S$ the set of pairwise sums, $S + S = \{x + y \mid x, y \in S\}$. A set S is said to be *sum-free* if $S \cap (S + S) = \emptyset$, that is if there do not exist $x, y, z \in S$ for which $x + y = z$.

We call a sum-free set *complete* if there is an n_0 such that for all $n > n_0$, $n \in S \cup (S+S)$, that is either $n \in S$ or there exist $x, y \in S$ with

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[1] It seems that the problem as stated by Guy doesn't appear in the reference cited.

$x + y = n$. Equivalently, S is complete if and only if it is constructed greedily from a finite set.

We call S *ultimately periodic with period* m if its characteristic function is ultimately periodic with period m, that is if there exists a positive integer n_o such that for all $n > n_0$, $n \in S$ if and only if $n + m \in S$: otherwise we say that S is *aperiodic*.

The following sets S_α are a natural set of aperiodic sum-free sets: for α irrational, define

$$S_\alpha = \left\{ n | \{n\alpha\} \in (\frac{1}{3}, \frac{2}{3}) \right\}$$

where $\{x\}$ denotes the fractional part of x.

Clearly, S_α is aperiodic: indeed, if S_α were ultimately periodic with period m, then for sufficiently large k, $km \notin S_\alpha$. However, since $m\alpha$ is irrational, $\{km\alpha\}$ is dense on $(0, 1)$ as k runs through the positive integers.

Since the sets S_α all have density $\frac{1}{3}$ and it is easily seen that $S_\alpha \cup (S_\alpha + S_\alpha)$ has density 1, it might be hoped that amongst the S_α we might find a complete set, thus answering Dickson's question in the negative (it is clear that not every S_α can be complete, since the set of complete sum-free sets is countable, and there are uncountably many S_α). We prove below that this is not the case.

3. Preliminaries

We need an understanding of when $n \notin S_\alpha \cup (S_\alpha + S_\alpha)$.

Lemma 1. *$n \in S_\alpha + S_\alpha$ if and only if either*
(i) $\{n\alpha\} = \frac{1}{3} - \epsilon$, where $0 < \epsilon < \frac{1}{3}$ and there exists an $m < n$ with $\frac{2}{3} - \{m\alpha\} = \delta < \epsilon$
or
(ii) $\{n\alpha\} = \frac{2}{3} + \epsilon$, where $0 < \epsilon < \frac{1}{3}$ and there exists an $m < n$ with $\{m\alpha\} - \frac{1}{3} = \delta < \epsilon$.
Further, we may choose $m \leq \frac{n}{2}$ or $\delta \leq \frac{\epsilon}{2}$.

Proof: Suppose that $n \in S_\alpha + S_\alpha$. Then either $\{n\alpha\} \in (0, \frac{1}{3})$ or $\{n\alpha\} \in (\frac{2}{3}, 1)$. Suppose the former. Let $m_1, m_2 \in S_\alpha$, with $m_1 + m_2 = n$, and

$$\{m_1\alpha\} = \frac{2}{3} - \delta_1, \quad \text{and} \quad \{m_2\alpha\} = \frac{2}{3} - \delta_2.$$

Then $\{(m_1 + m_2)\alpha\} = \{\frac{4}{3} - \delta_1 - \delta_2\} = \{\frac{1}{3} - (\delta_1 + \delta_2)\}$ and since $\{n\alpha\} = \frac{1}{3} - \epsilon$, we have $\delta_1 + \delta_2 = \epsilon$. Since $m_1 + m_2 = n$, the smaller of m_1 and m_2 is at most $\frac{n}{2}$, and since $\delta_1 + \delta_2 = \epsilon$, the smaller of δ_1 and δ_2 is at most $\frac{\epsilon}{2}$.

Conversely, if there is an m with $\frac{2}{3} - \{m\alpha\} = \delta < \epsilon$ then clearly $(n-m) \in S_\alpha$, and so $n \in S_\alpha + S_\alpha$.

The case where $\{n\alpha\} \in (\frac{2}{3}, 1)$ is handled similarly.

We also require the following basic facts about continued fractions (cf [7]):

Lemma 2. *If θ is irrational, with continued fraction convergents $\frac{a_i}{b_i}$ and continued fraction expansion*

$$q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cfrac{1}{q_4 + \dots}}}}$$

then for every i,
(i) $\frac{a_{2i}}{b_{21}} < \theta < \frac{a_{2i+1}}{b_{2i+1}}$
(ii) $a_i b_{i+1} - a_{i+1} b_i = \pm 1$
(iii) $a_{i+1} = q_i a_i + a_{i-1}$, $b_{i+1} = q_i b_i + b_{i-1}$
(iv) $\left|\frac{a_i}{b_i} - \theta\right| < \frac{1}{b_i b_{i+1}}$
(v) If $a, b \in \mathbb{Z}$ with $0 < b < b_{n+1}$, and $b \neq b_n$, then $|a - b\theta| > |a_n - b_n\theta|$.

4. The Result

Theorem 1. *For every irrational α, the set S_α is incomplete.*

Proof: Consider the continued fraction convergents $\frac{a_i}{b_i}$ to $\beta = 3\alpha$. Suppose i is even: then

$$\frac{a_i}{b_i} < \beta, \text{ so } b_i\beta > a_i.$$

Hence if $a_i \equiv 2 \pmod 3$, then $0 < b_i\beta - a_i < \frac{1}{b_{i+1}}$ so

$$0 < \{b_i\alpha\} - \frac{2}{3} < \frac{1}{3b_{i+1}},$$

and by Lemma 2(v), we see that there are no a, b, with $b < b_i$ and $|b\beta - a| < |b_i\beta - a_i|$. Hence $b_i \notin S_\alpha \cup (S_\alpha + S_\alpha)$.

Similarly, if i is odd, and $a \equiv 1 \pmod 3$ then

$$\frac{a_i}{b_i} > \beta, \text{ so } b_i\beta < a_i$$

and

$$0 < \frac{1}{3} - \{b_i\alpha\} < \frac{1}{3b_{i+1}}$$

and so $b_i \notin S_\alpha \cup (S_\alpha + S_\alpha)$.

In the remainder of the proof we may thus assume that for all i sufficiently large,

$$a_{2i} \not\equiv 2 \pmod 3 \text{ and } a_{2i+1} \not\equiv 1 \pmod 3.$$

Further, since $a_i b_{i+1} - a_{i+1} b_i = \pm 1$, we have $(a_i, a_{i+1}) = 1$, and so a_i, a_{i+1} cannot both be divisible by 3.

Hence one of the following situations occurs:
(i) $a_{2i} \equiv 1 \pmod 3$, $a_{2i+1} \equiv 2 \pmod 3$ for infinitely many i,
(ii) $a_{2i-1} \equiv 0 \pmod 3$, $a_{2i} \equiv 1 \pmod 3$ for infinitely many i,
(i) $a_{2i} \equiv 0 \pmod 3$, $a_{2i+1} \equiv 2 \pmod 3$ for infinitely many i.
Case (i): We will show that $b_{2i+1} - b_{2i} \notin S_\alpha \cup (S_\alpha + S_\alpha)$.
Let

$$\{b_{2i}\alpha\} = \frac{1}{3} + \epsilon_{2i}$$

$$\{b_{2i+1}\alpha\} = \frac{2}{3} - \epsilon_{2i+1}.$$

Then

$$\{(b_{2i+1} - b_{2i})\alpha\} = \frac{1}{3} - \epsilon_{2i+1} - \epsilon_{2i},$$

so $b_{2i+1} - b_{2i} \notin S_\alpha$.
Further, for every a, b with $1 \leq b < b_{2i+1}, b \neq b_{2i}$ we have

$$\begin{aligned}\left|\frac{2}{3} - \{b\alpha\}\right| &\geq \left|b\alpha - \frac{a}{3}\right| \\ &= \frac{1}{3}\left|b\beta - a\right| \\ &> \frac{1}{3}\left|b_{2i}\beta - a_{2i}\right| \\ &= \epsilon_{2i} \\ &> \frac{1}{2}(\epsilon_{2i+1} + \epsilon_{2i})\end{aligned}$$

Thus there is no $b < b_{2i+1} - b_{2i}$ with

$$\frac{2}{3} - \frac{1}{2}(\epsilon_{2i+1} + \epsilon_{2i}) < \{b\alpha\} < \frac{2}{3}.$$

Hence by Lemma 1, $b_{2i+1} - b_{2i} \notin S_\alpha + S_\alpha$.
Case (ii): we will show that $b_{2i} + b_{2i-1} \notin S_\alpha \cup (S_\alpha + S_\alpha)$
Let

$$\{b_{2i-1}\alpha\} = 1 - \epsilon_{2i-1}$$

$$\{b_{2i}\alpha\} = \frac{1}{3} + \epsilon_{2i}.$$

Then

$$\{(b_{2i-1}+b_{2i})\alpha\} = \frac{1}{3} - (\epsilon_{2i-1} - \epsilon_{2i})$$

so $b_{2i} + b_{2i-1} \notin S_\alpha$. Further, for every a, b with $1 \leq b < b_{2i}, b \neq b_{2i-1}$ we have

$$\begin{aligned} \left|\{b\alpha\} - \frac{2}{3}\right| &\geq \left|b\alpha - \frac{a}{3}\right| \\ &= \frac{1}{3}\left|b\beta - a\right| \\ &> \frac{1}{3}\left|b_{2i-1}\beta - a_{2i-1}\right| \\ &= \epsilon_{2i-1}. \end{aligned}$$

Now, since $\frac{1}{2}(b_{2i} + b_{2i-1}) < b_{2i}$ there is no $b < \frac{1}{2}(b_{2i} + b_{2i-1})$ with

$$\frac{2}{3} - (\epsilon_{2i-1} - \epsilon_{2i}) < \{b\alpha\} < \frac{2}{3}.$$

Hence, by Lemma 1, $b_{2i-1} + b_{2i} \notin S_\alpha + S_\alpha$.
Case (iii) is proved in a similar fashion to case (ii), obtaining $b_{2i}+b_{2i+1} \notin S_\alpha \cup (S_\alpha + S_\alpha)$, completing the proof of the theorem.

5. Open Questions

In this section we present some open questions in related areas.

(1) The sets considered above show that there are aperiodic maximal sum-free sets for which $S \cup (S + S)$ omits about $\log n$ elements up to n. Are there maximal aperiodic sum-free sets which omit $o(\log n)$ elements?
(2) If there are no aperiodic complete sum-free sets, then there is a function $M = M(n)$ so that if S is a sum-free set, and if $S \cup (S + S)$ contains all integers greater than n, then S has period $\leq M$. How fast must $M(n)$ grow?
(3) Erdös [5] and Alon and Kleitman [1] have shown using similar ideas that if S is a set of n integers, then there is a sum-free subset of S of cardinality greater than $\frac{n}{3}$. Is $\frac{n}{3}$ best possible?
(4) By considering α chosen uniformly at random from the interval $(\frac{1}{m^2-1}, \frac{m}{m^2-1})$ we see that any set of S of n positive integers contains a subset of cardinality $\frac{n}{m+1}$ with the property that it has no solutions to the equation $x_1 + x_2 + \cdots + x_m = y$. Is this best possible? In particular, if c_m is such that any set S of cardinality n contains a subset of cardinality $c_m n$ with no

solutions to $x_1 + x_2 + \cdots + x_m = y$ then what is the behaviour of c_m as $m \to \infty$? Does $c_m \to 1$? To 0?

(5) Similarly, we see that if $S = \{s_1, s_2, s_3, \ldots\}$ then there is a subset $S' = \{s_{i_1}, s_{i_2}, s_{i_3}, \ldots\}$ having no solution to $x_1 + x_2 + \cdots + x_m = y$ for which

$$\lim_{n\to\infty} \frac{i_n}{n} = \frac{1}{m+1}.$$

Is $\frac{1}{m+1}$ best possible here?

(6) Call a set *strongly* m-sum-free if it contains no solution to $x_1 + x_2 + \cdots + x_j = y$ for $2 \le j \le m$. Given a set S of n positive integers, how large a strongly m-sum-free set must it contain? Let d_m be such that S must contain a set of size $d_m n$. What is the behaviour of d_m as $m \to \infty$? Is it possible that d_m is bounded away from 0?

(7) Which sets S having no solutions to $x_1 + x_2 + \cdots + x_m = y$ have the greatest density? In particular, if k is the least integer not dividing $m - 1$ then the set of integers congruent to $1 \pmod k$ has no solutions. Is this best possible? This is easy to show when $m = 2$, and has been shown by Łuczak (personal communication) in the case $m = 3$. The first interesting case is thus $m = 7$. (Note added in proof: it appears (personal communication) that Schoen has shown that this is best possible).

(8) Cameron [4] has considered a probability measure on the set of all sum-free sets: does $\{S | \exists \alpha \text{ s.t. } S \subseteq S_\alpha\}$ have positive measure?

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