Slow Convergence of a Double-Biased Sequence

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Abstract

We consider a sequence, p_n, defined by

$$p_0 = 0; \quad p_1 = 1; \quad p_n = \frac{1}{2} \cdot p_{\lfloor \frac{n}{3} \rfloor} + \frac{1}{2} \cdot p_{\lfloor \frac{n}{2} \rfloor}, \quad n \geq 2.$$

This is a simple example of a sequence in which the n^{th} term is a weighted average of the preceding terms and the weights for p_n are heavily concentrated at two previous elements in the sequence. It is known [1] that if the weights for the n^{th} term of a sequence are sharply concentrated around a single previous element and if the sequence appears to be oscillating at the beginning, then the sequence will continue to oscillate, and, hence, will not converge. Although the double-biased sequence which we consider appears to be oscillating at the beginning, we show that it does, in fact, slowly converge. Specifically, we prove that

$$p_n \to \frac{2}{1 + \log_2 3}.$$

1 Introduction

Lampert and Slater [3] posed the following question. Begin with a complete graph on n vertices, and repeat the following "knockout" procedure while there remain two or more vertices. Each remaining vertex chooses a neighbor at random to be knocked out of the graph. The process terminates when there is either a single vertex or no vertex remaining. The question is, as a function of n, does q_n (the probability that one vertex remains) tend to a limit as $n \to \infty$?

In their analysis of this problem, Calkin, Canfield and Wilf [1] considered the limiting behavior of a sequence of real numbers q_n in the interval $[0,1]$ defined by $q_0 = 0$, $q_1 = 1$, and, for $n \geq 2$, q_n equals a weighted average of preceding terms in the sequence. They proved that if the weights are sharply concentrated around a value αn, where α is fixed in $[0,1]$, and if this sequence oscillates according to certain criteria up to a computable point, then it will continue to oscillate and, hence, will not converge. The key requirement in their argument is that the weights for the n^{th} term be sharply concentrated around a single previous element in the sequence.

We consider a sequence, p_n, where the weights for p_n are heavily concentrated at *two* previous elements in the sequence. Although the sequence appears to be oscillating at the beginning, it does, in fact, slowly converge. The following game gives rise to this sequence.

Double-Bias Game

A player begins the game with n marbles. At each turn he flips a fair coin. If it is heads, then he discards $\lceil \frac{n}{2} \rceil$ of the marbles; if it is tails, then he discards $\lceil \frac{2n}{3} \rceil$ of the marbles. Play stops when the player has either one or zero marbles remaining.

What is the probability that a player finishes with one marble given that he begins with n marbles, for large n (*i.e.*, what are the chances that a player does not lose all his marbles)?

2 The Main Results

We prove the following results.

Theorem 1

$$p_n = \sum_{b=0}^{\lfloor \log_3 n \rfloor} \binom{a+b}{b} 2^{-(a+b)}, \quad \text{where } a = \lfloor \log_2 n - b \log_2 3 \rfloor.$$

Theorem 2

$$\lim_{n\to\infty} p_n = \frac{2}{1+\log_2 3}.$$

Proof of Theorem 1.

Clearly,

$$\begin{aligned} p_0 &= 0; \\ p_1 &= 1; \\ p_n &= \frac{1}{2} \cdot p_{\lfloor \frac{n}{3} \rfloor} + \frac{1}{2} \cdot p_{\lfloor \frac{n}{2} \rfloor}, \quad n \geq 2. \end{aligned}$$

$$
\begin{aligned}
P(x) &= \sum_{n\geq 0} p_n x^n \\
&= x + \frac{1}{2}\cdot \sum_{n\geq 2} (p_{\lfloor \frac{n}{3} \rfloor} + p_{\lfloor \frac{n}{2} \rfloor}) x^n \\
&= x + \frac{1}{2}\left[\sum_{n\geq 2} p_{\lfloor \frac{n}{3} \rfloor} x^n + \sum_{n\geq 2} p_{\lfloor \frac{n}{2} \rfloor} x^n \right] \\
&= x + \frac{1}{2}\left[\sum_{\substack{n\geq 2,\\ n\equiv 0 \pmod 2}} p_{\frac{n}{2}} x^n + \sum_{\substack{n\geq 2,\\ n\equiv 1 \pmod 2}} p_{\frac{n-1}{2}} x^n \right. \\
&\quad \left. + \sum_{\substack{n\geq 2,\\ n\equiv 0 \pmod 3}} p_{\frac{n}{3}} x^n + \sum_{\substack{n\geq 2,\\ n\equiv 1 \pmod 3}} p_{\frac{n-1}{3}} x^n + \sum_{\substack{n\geq 2,\\ n\equiv 2 \pmod 3}} p_{\frac{n-2}{3}} x^n \right] \\
&= x + \frac{1}{2}\left[\sum_{\substack{n\geq 2,\\ n \text{ even}}} p_{\frac{n}{2}} (x^2)^{n/2} + x \sum_{\substack{n\geq 2,\\ n \text{ odd}}} p_{\frac{n-1}{2}} (x^2)^{(n-1)/2} \right. \\
&\quad \left. + \sum_{\substack{n\geq 2,\\ 3|n}} p_{\frac{n}{3}} + x \sum_{\substack{n\geq 2,\\ 3|(n-1)}} p_{\frac{n-1}{3}} (x^3)^{\frac{n-1}{3}} + x^2 \sum_{\substack{n\geq 2,\\ 3|(n-2)}} p_{\frac{n-2}{3}} (x^3)^{\frac{n-2}{3}} \right] \\
&= x + \frac{1}{2}\cdot \frac{1-x^2}{1-x}\cdot P(x^2) + \frac{1}{2}\cdot \frac{1-x^3}{1-x}\cdot P(x^3)
\end{aligned}
$$

Iterating this functional equation, we see that

$$
\begin{aligned}
P(x) &= \frac{1}{1-x} \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} \binom{a+b}{b} 2^{-(a+b)} x^{2^a 3^b} (1 - x^{2^a 3^b}) \\
&= \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} \binom{a+b}{b} 2^{-(a+b)} x^{2^a 3^b} \sum_{r=0}^{2^a 3^b - 1} x^r .
\end{aligned}
$$

So $p_n = [x^n]P(x) = \sum_{a,b} \binom{a+b}{b} 2^{-(a+b)}$, summed over all a, b for which $\frac{n}{2} < 2^a 3^b \leq n$. Clearly, $0 \leq b \leq \log_3 n$, and, for each b in this range, there is exactly one value that a can take. Specifically, $a = \lfloor \log_2 n - b \log_2 3 \rfloor$. Finally, we have

$$
p_n = \sum_{b=0}^{\lfloor \log_3 n \rfloor} \binom{a+b}{b} 2^{-(a+b)}, \quad \text{where } a = \lfloor \log_2 n - b \log_2 3 \rfloor .
$$

□

The standard approach to estimating such a sum is to find the maximum term and to estimate the ratio of each term to it. It is easily seen (*e.g.*, by considering ratios of consecutive terms) that the largest term in the sum occurs when a and b are about equal. In fact, the maximum (maxima) must occur when $a = b = k$ or at $b = k$, $a = k \pm 1$. For simplicity, we shall compare all terms in the sum to $\binom{2k}{k}2^{-2k}$, where $k = \lfloor \log_6 n \rfloor$. This is of the same order of magnitude as the largest term in the sum.

To estimate the sum, we rewrite the sum so as to center the terms around this value k. Specifically, let $b = k + t$, and let $a = k - x$, where $x = x(t) = t \cdot \log_2 3 + \epsilon(t)$, $\quad |\epsilon(t)| < 1$ is determined by the formula for a. The new sum is

$$\sum_t \binom{2k + t - x}{k + t} \cdot 2^{-(2k+t-x)}.$$

In the proof of Theorem 2, we shall estimate this sum by dividing each term by $\binom{2k}{k}2^{-2k}$. First, we collect some standard results.

Lemma 3 *Suppose* $t = o(k^{2/3})$, *then*

$$\prod_{j=1}^{t} \left(1 - \frac{j}{k}\right) = \exp\left(\frac{-t^2}{2k}\right) \cdot (1 + o(1)).$$

Proof .

See, for example, [4].

□

Lemma 4

$$\sum_{t=-\infty}^{\infty} \exp\left(\frac{-A}{k}t^2\right) = \sqrt{\frac{k\pi}{A}} + o(1), \quad \textit{as } k \to \infty.$$

Proof .

See, for example, [2].

□

Lemma 5

$$\sum_{t \geq k^{3/5}} \exp\left(\frac{-A}{k}t^2\right) = o(1).$$

Proof .

See, for example, [4].

□

Lemma 6

$$\sum_{|t|>k^{3/5}} \binom{2k+t-x}{k+t} \cdot 2^{-(2k+t-x)} = o(1).$$

Proof .
See, for example, [4].

□

Proof of Theorem 2.
Stirling's Approximation implies that

$$\binom{2k}{k} \sim 2^{2k} \cdot \frac{1}{\sqrt{\pi k}}.$$

Also,

$$\frac{\binom{2k+t-x}{k+t}}{\binom{2k}{k}} = 2^{t-x} \cdot \frac{1 \cdot (1-\frac{1}{k}) \cdot \cdots \cdot (1-\frac{x-1}{k})}{1 \cdot (1-\frac{1}{2k}) \cdot \cdots \cdot (1-\frac{x-t-1}{2k}) \cdot (1+\frac{1}{k}) \cdot \cdots \cdot (1+\frac{t}{k})}.$$

From Lemma 3, provided $t = o(k^{2/3})$, we know that

$$\begin{aligned} \prod_{j=1}^{x-1}\left(1-\frac{j}{k}\right) &= \exp\left(\frac{-(x-1)^2}{2k}\right) \cdot (1+o(1)), \\ \prod_{j=1}^{x-t-1}\left(1-\frac{j}{2k}\right) &= \exp\left(\frac{-(x-t-1)^2}{4k}\right) \cdot (1+o(1)), \text{ and} \\ \prod_{j=1}^{t}\left(1+\frac{j}{k}\right) &= \exp\left(\frac{t^2}{2k}\right) \cdot (1+o(1)). \end{aligned}$$

Thus,

$$\frac{\binom{2k+t-x}{k+t}}{\binom{2k}{k}} = 2^{t-x} \cdot \exp\left(\frac{2x-x^2-t^2-2tx+2t-1}{4k}\right) \cdot (1+o(1)).$$

By substituting $x = t \cdot \log_2 3 + \epsilon(t)$, with $|\epsilon| < 1$, we have

$$\exp\left(\frac{2x-x^2-t^2-2tx+2t-1}{4k}\right) = \exp\left(-A\frac{t^2}{k} + B(1-\epsilon)\frac{t}{k} + \frac{O(1)}{k}\right),$$

where $B = \frac{1+\log_2 3}{2}, \quad A = B^2.$

So we now have

$$\begin{aligned}
\frac{\binom{2k+t-x}{k+t}}{\binom{2k}{k}} &= 2^{t-x} \cdot \exp\left(-A\frac{t^2}{k} + O(1)\frac{t}{k}\right) \cdot (1+o(1)) \\
&= 2^{t-x} \cdot \exp\left(-A\frac{t^2}{k} + o(1)\right) \cdot (1+o(1)), \text{ since } t = o(k^{2/3}), \\
&= 2^{t-x} \cdot \exp\left(-A\frac{t^2}{k}\right) \cdot (1+o(1)).
\end{aligned}$$

Then, if $LHS = \sum_{|t|<k^{3/5}} \binom{2k+t-x}{k+t} \cdot 2^{-(2k+t-x)}$, we have

$$\begin{aligned}
LHS &= \frac{2^{2k}}{\sqrt{\pi k}} \sum_{|t|<k^{3/5}} 2^{t-x} \exp\left(-A\frac{t^2}{k}\right) 2^{-2k-t+x} \cdot (1+o(1)) \\
&= \frac{1}{\sqrt{\pi k}} \sum_{|t|<k^{3/5}} \exp\left(-A\frac{t^2}{k}\right) \cdot (1+o(1)) \\
&= \frac{1}{\sqrt{\pi k}} \left(\sum_{t\in\mathbb{Z}} \exp\left(-A\frac{t^2}{k}\right) - \sum_{|t|\geq k^{3/5}} \exp\left(-A\frac{t^2}{k}\right)\right) \cdot (1+o(1)) \\
&= \frac{1}{\sqrt{\pi k}} \left(\sqrt{\frac{k\pi}{A}}(1+o(1))\right), \text{ by Lemmas 4 and 5,} \\
&= \frac{1}{\sqrt{A}}(1+o(1)) \\
&= \frac{2}{1+\log_2 3}, \quad \text{as } k \to \infty.
\end{aligned}$$

Finally, by Lemma 6, the result follows.

□

3 Comparison with numerical results

More careful analysis easily shows that the errors in the estimation give

$$\begin{aligned}
p_n &= \frac{2}{1+\log_2 3} + O\left(\frac{1}{\sqrt{k}}\right) \\
&= \frac{2}{1+\log_2 3} + O\left(\frac{1}{\sqrt{\log_6 n}}\right).
\end{aligned}$$

Computational results indicate that the error term is probably closer to $\frac{1}{\log_6 n}$ rather than $\frac{1}{\sqrt{\log_6 n}}$. To illustrate this we give the following results.

n	$\frac{1}{\sqrt{\log_6 n}}$	$\frac{1}{\log_6 n}$	$p_n - \frac{2}{1+\log_2 3}$
10^5	3.945×10^{-1}	1.556×10^{-1}	-1.859×10^{-2}
10^{50}	1.248×10^{-1}	1.556×10^{-2}	1.003×10^{-3}
10^{500}	3.945×10^{-2}	1.556×10^{-3}	-2.158×10^{-4}
10^{5000}	1.248×10^{-2}	1.556×10^{-4}	1.639×10^{-5}
10^{50000}	3.945×10^{-3}	1.556×10^{-5}	-4.170×10^{-7}
10^{98000}	2.818×10^{-3}	7.940×10^{-6}	-1.340×10^{-6}

4 Generalizations and Open Problems

Following the proof above, we can generalize our result to a weighted double-biased scenario.

Theorem 7 *If $p_0 = 0$, $p_1 = 1$ and, for $n \geq 2$ and $0 < q < 1$, $p_n = q \cdot p_{\lfloor \frac{n}{3} \rfloor} + (1-q) \cdot p_{\lfloor \frac{n}{2} \rfloor}$, then*

$$\lim_{n\to\infty} p_n = \frac{1}{(1-q) + q \cdot \log_2 3}.$$

Two natural generalizations still remain to be studied. First, we would like to characterize the double-biased case in which the weights are at locations other than 1/3 and 1/2; that is, $p_n = q \cdot p_{\lfloor \alpha n \rfloor} + (1-q) \cdot p_{\lfloor \beta n \rfloor}, \quad 0 < \alpha, \beta < 1$. It is not at all clear whether the sequence will converge even for other very simple cases. However, we hope to be able to use the methods in our current proof for $\alpha, \beta \in \mathbb{Q}$, but we note that an entirely new proof is needed if α or β is irrational.

Finally, we would like to explore the nature of the triple-biased (or higher) case.

References

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