On the expected performance of a parallel algorithm for finding maximal independent subsets of a random graph

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Abstract

We consider the parallel greedy algorithm of Coppersmith, Raghavan and Tompa [CRT] for finding the lexicographically first maximal independent set of a graph. We prove an $\Omega(\log n)$ bound on the expected number of iterations for most edge densities. This complements the $O(\log n)$ bound proved in Calkin and Frieze [CF].

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1 Introduction

In this note we consider the problem of finding the lexicographically first maximal independent set (LFMIS) in a random graph. Coppersmith, Raghavan and Tompa [CRT] describe a parallel version of the standard greedy algorithm for this problem:
Suppose we are given a graph $G = (V, E)$, $V = [n] = \{1, 2, \ldots, n\}$. For $Z \subseteq V$ we let

$$\Gamma^+(Z) = \{x \notin Z \ : \ xz \in E \text{ for some } z < x,\, z \in Z\},$$

and

$$\Gamma^-(Z) = \{x \notin Z \ : \ xz \in E \text{ for some } z > x,\, z \in Z\}.$$

Note that we have implicitly oriented the edges from low to high.

algorithm PARALLEL GREEDY (G);
    begin
        GIS $\leftarrow \emptyset$;
        until G has no vertices **do**
            begin
                let $S = \{a : \Gamma^-(a) = \emptyset\}$;
                GIS $\leftarrow$ GIS$\cup S$;
                remove $S \cup \Gamma(S)$ from G
            end
        output GIS
    end

It is easy to see ([CRT], Lemma 2.1) that GIS is the LFMIS. Cook [C] showed that the problem of computing the LFMIS of a graph is complete for P and so is not in NC unless NC=P. PARALLEL–GREEDY can be implemented on a CRCW PRAM in $O(1)$ time per iteration if one processor is allocated to each edge of G.

Coppersmith, Raghavan and Tompa showed that if $T(n, p)$ denotes the *expected* number of iterations $\tau = \tau(G)$ when $G = G_{n,p}$ then $T(n, p) = O(\frac{(\log n)^2}{\log \log n})$. ($G_{n,p}$ is the random graph with vertex set $[n]$ where each edge occurs independently with probability $p = p(n)$.).

They conjectured that $T(n, p) = O(\log n)$ and subsequently Calkin and Frieze [CF] proved

Theorem 1
(a) $\frac{\alpha \log n}{4 \log\log n} \leq T(n,p)$ *for* $\frac{1}{n} \leq p \leq \frac{1}{n^{\alpha}}$ *where* $0 < \alpha \leq 1$ *is constant*
(b) $T(n,p) = O(\log n)$.
The hidden constant in (b) is independent of p.

Note that our inequalities are only claimed for n large.

The upper bounds and lower bounds in Theorem 1 are slightly different. It leaves open the possibility that $T(n,p) = O(\frac{\log n}{\log\log n})$ throughout. The aim of this paper is to shed more light on this problem, and to prove

Theorem 2 *Assume* $0 \leq \alpha < 1$, α *constant.*
(a) $T(n,p) \leq \frac{3 \log n}{(1-\alpha)\log\log n}$ *for* $p \leq \frac{(\log n)^{\alpha}}{n}$,
(b) $T(n,p) = \Omega(\log n)$ *for* $\alpha \geq p \geq \frac{1}{n^{\alpha}}$,
where the hidden constant in (b) depends on α.

Proof:
(a) Let $G = G_1 \supseteq G_2 \supseteq G_3 \supseteq \ldots$ denote the sequence of graphs produced by each iteration of the algorithm.

For $v \in V(G_t)$ and $t \geq 1$ let $\alpha(t,v)$ = the length of the longest directed path in G_t which ends at v (a path $(v_1,\ v_2, \ldots v_k$, is directed if $v_1 < v_2 < \ldots v_k$.)

Clearly, if $v \in V(G_{t+1})$ then $\alpha(t+1,v) \leq \alpha(t,v) - 2$.

Hence

$$\tau(G) \leq \frac{1}{2} \max\{v \in V(G) : \alpha(1,v)\}.$$

Thus

$$\begin{aligned}
\Pr(\tau(G_{n,p}) \geq k) &\leq \text{E}(\#\text{ of directed paths of length } 2k) \\
&= \binom{n}{2k} p^{2k-1} \\
&\leq n \left(\frac{nep}{2k}\right)^{2k-1} \\
&\leq n \left(\frac{e(\log n)^{\alpha}}{2k}\right)^{2k-1}.
\end{aligned}$$

Hence, with $k_0 = \lceil \frac{2\log n}{(1-\alpha)\log\log n} \rceil$,

$$\begin{aligned} T(n,p) &= \sum_{k=1}^{n} \Pr(\tau(G_{n,p}) \geq k) \\ &\leq k_0 + n \sum_{k=k_0+1}^{n} \left(\frac{e(\log n)^\alpha}{2k} \right)^{2k_0-1} \\ &\leq k_0 + 2n \left(\frac{e(\log n)^\alpha}{2k_0} \right)^{2k_0-1} \\ &\leq k_0 + 2n \left(\frac{A\log\log n}{(\log n)^{1-\alpha}} \right)^{2k_0-1} \end{aligned}$$

where $A = e(1-\alpha)/4$,

$$= k_0 + o(1).$$

This completes the proof of (a).
(b) This is somewhatless trivial.
Let

$$\begin{aligned} V_t &= V(G_t) \\ &= \{ \text{ vertices remaining at the start of round } t \} \\ S_t &= \text{Set } S \text{ found in round } t \\ &= \{ \text{ sources found in round } t \} \ , \\ N_t &= \Gamma(S_t) \cap V_t \\ &= \{ \text{ neighbours of } S_t \text{ deleted in round } t \ \}. \end{aligned}$$

Suppose $i \geq 2$ and A_t, B_t, $1 \leq t \leq i-1$ is some disjoint collection of subsets of V. Then we have $S_t = A_t$, $N_t = B_t$ for $1 \leq t \leq i-1$ if and only if
(2a) $v \in A_t$ implies $\Gamma^-(v) \subseteq \bigcup_{s=1}^{t-1} B_s$ and $\Gamma^-(v) \cap B_{t-1} \neq \emptyset$, $1 \leq t \leq i-1$ (when $t=1$, drop the second condition)
(2b) $v \in B_t$ implies $\Gamma^-(v) \cap \bigcup_{s=1}^{t-1} A_s = \emptyset$ and $\Gamma^-(v) \cap A_t \neq \emptyset$, $1 \leq t \leq i-1$
and

$$v \in C = V \ - \ \bigcup_{t=1}^{i-1} (A_t \cup B_t) \text{ implies}$$

(3a) $\Gamma^-(v) \cap \bigcup_{t=1}^{i-1} A_t = \emptyset$,
(3b) $\Gamma^-(v) \cap (B_{i-1} \cup C) \neq \emptyset$.

Suppose now that we choose sets A_t, B_t, $1 \le t \le i-1$ satisfying (2) and condition on the event

$$\mathcal{E} = \{S_t = A_t,\ N_t = B_t,\ V_i = C :\ 1 \le t \le i-1\}.$$

It is important to establish the conditional distribution of the sets $\Gamma_i^-(v) = \Gamma^-(v) \cap V_i$, $v \in V_i$, $i \ge 2$. For $v \in V_i$ let $R_v^i = [v-1] \cap (V_i \cup B_{i-1})$ and $r_v = |R_v^i|$.
Claim 1
(i) The sets $\Gamma_i^-(v)$, $v \in V_i$ are stochastically independent,
(ii) $\Gamma_i^-(v)$ is a random subset of R_v^i chosen through r_v Bernoulli trials conditioned on the occurence of at least one success, i. e.
(4) $\Pr(|\Gamma_i^-(v)| \ = k) = \binom{r_v}{k} p^k (1-p)^{r_v-k}/(1-(1-p)^{r_v})$, $1 \le k \le r_v$
and each k-subset is equally likely.
Proof (of Claim) To prove (i) simply observe that condition (3) on $v \in C$ only involves edges directed into v, and that the conditions in (2) only involve edges directed into $V - C$.

Now consider (ii). $v \in V_2$ if and only if $\Gamma_i^-(v) \ne \emptyset$ and $\Gamma_i^-(v) \cap S_1 = \emptyset$ and these conditions are equivalent to (ii). We can now proceed inductively. Fix $v \in V_i$. If $v \notin S_i \cup N_i$ then we learn (a) $\Gamma_i^-(v) \cap V_i \ne \emptyset$, then (ii) $\Gamma_i^-(v) \cap S_i = \emptyset$ and so finally that

$$\Gamma_i^-(v) \cap (V_i - S_i) = \Gamma_i^-(v) \cap R_v^{i+1} \ne \emptyset.$$

Thus (4) continues to hold.
End of proof (of claim). We now continue with the proof of our Theorem. Choose β, $\alpha < \beta < 1$. Now choose $i \le \tau = \lceil \frac{(1-\alpha)\log n}{10} \rceil$ and assume that $V_i = \{x_1 < x_2 < \ldots < x_s\}$. Partition V_i into X_1, X_2, Y where $X_1 = \{x_1, x_2, \ldots x_a\}$, $a = \lceil \log n/p \rceil$, $X_2 = \{x_{a+1}, x_{a+2}, \ldots x_b\}$, $b = \lceil (\log n)^2/p \rceil$, and Y is the rest of V_i. We will show that a good proportion of Y is likely to remain in V_{i+1}, when V_i is large enough so that the above partition is actually possible.

Observe first that the proof of Claim 1 implies that if $r = |B_{i-1} \cap [x_j - 1]|$ then
(5) $\Pr(x = x_j \in S_i) = (1-(1-p)^r)(1-p)^{j-1}/(1-(1-p)^{r_x})$
$\le (1-p)^{j-1}.$
(At least one success is required in the r trials corresponding to $B_{i-1} \cap [x_j - 1]$ and no further successes.)
So if $\mathcal{A}_i = \{S_i \cap (X_2 \cup Y) = \emptyset\}$ then

(6) $\Pr(\bar{\mathcal{A}}_i) \leq \sum_{j>a}(1-p)^{j-1} = \frac{(1-p)^a}{p} \leq \frac{1}{np}$.

Let

$$\mathcal{B}_i = \{\Gamma^-(y) \cap X_2 \neq \emptyset, \forall y \in Y\}$$

It follows from Claim 1(ii) that if $y \in Y$ then

$$\begin{aligned} \Pr(\Gamma^-(y) \cap X_2 = \emptyset) &\leq (1-p)^{b-a} \\ &\leq n^{-(1-o(1))\log n} \end{aligned}$$

and so

(7) $\Pr(\bar{\mathcal{B}}_i) \leq n^{-(1-o(1))\log n}$.

Note that (6), (7) can be taken as true even if $Y = \emptyset$.

Let us now consider the size of S_i. Let $\delta_j = 1$ if $x_j \in S_i$ and $\delta_j = 0$ otherwise. It follows from Claim 1(i) that $\delta_1, \delta_2, \ldots, \delta_s$ are independent random variables. Also

$$\begin{aligned} E(|S_i|) &= \sum_{j=1}^{s} \Pr(\delta_j = 1) \\ &\leq \sum_{j=1}^{s}(1-p)^{j-1} \\ &\leq \frac{1}{p}. \end{aligned}$$

Note that we have $\Pr(\delta_j = 1) \leq (1-p)^{j-1}$ regardless of the history of the algorithm to this point. It follows that $|S_1| + |S_2| + \ldots + |S_i|$ is dominated by the sum of independent random variables each of which is the sum of a large number of independent 0-1 random variables. It follows from Theorem 1 of Hoeffding [H] that if

$$\mathcal{C}_i = \{|S_1| + |S_2| + \ldots + |S_i| < \frac{(1-\alpha)\log n}{2p}\}$$

then

$$\Pr(\bar{\mathcal{C}}_i) \leq \left(\frac{2ei}{(1-\alpha)\log n}\right)^{(1-\alpha)\log n/2p}$$

(Hoeffding proves that if $Z_1, Z_2, \ldots, Z_m$ are independent random variables with $0 \le Z_j \le 1$, $j = 1, 2, \ldots, m$ and $E(Z_1 + Z_2 + \cdots + Z_m) = m\mu$ then

$$\Pr(Z_1 + Z_2 + \cdots + Z_m \ge m(\mu + t)) \le \left(\left(\frac{\mu}{\mu + t} \right)^{\mu + t} \left(\frac{1 - \mu}{1 - \mu - t} \right)^{1 - \mu - t} \right)^m .$$

So if $t = (\theta - 1)\mu$

$$\Pr(Z_1 + Z_2 + \cdots + Z_m \ge \theta m\mu) \le \left(\theta^{-\theta} e^{\theta - 1} \right)^{m\mu} < \left(\frac{e}{\theta} \right)^{\theta m\mu} .$$

We use this inequality with $m\mu = \frac{i}{p}$ and $\theta m\mu = \frac{(1-\alpha)\log n)}{2p}$.)

Note that $\mathcal{C}_\tau \subseteq \mathcal{C}_{\tau-1} \subseteq \cdots \subseteq \mathcal{C}_1$ and
(8) $\Pr(\bar{\mathcal{C}_\tau}) \le n^{-(1-\alpha)\log(5/e)/2\alpha}$.

Consider the size of $Y \cap V_{i+1}$. Using Claim 1(ii) we see that, given $\mathcal{A}_i \cap \mathcal{B}_i$, the edges joining X_1 to Y are unconditioned. So, by another use of [H],
(9) $\Pr(|V_{i+1}| \le \left(1 - \frac{1}{(\log n)^2}\right) |Y|(1-p)^{|S_i|} \mid \mathcal{A}_i \cap \mathcal{B}_i, \ |S_i|) \le \exp\left\{ -\frac{|Y|(1-p)^{|S_i|}}{2(\log n)^4} \right\}$
since if $y \in Y$ then $\Pr(y \in V_{i+1} \mid \mathcal{A}_i \cap \mathcal{B}_i, \ |S_i|) = (1-p)^{|S_i|}$.
Now let

$$\mathcal{D}_i = \left\{ |V_i| > \left(1 - \frac{2}{(\log n)^2}\right)^{i-1} n(1-p)^{|S_1| + |S_2| + \ldots + |S_{i-1}|} \right\} .$$

Then we have
(10) $\Pr(\bar{\mathcal{D}}_{i+1}) \le \ \Pr(\bar{\mathcal{A}_i} \cap \bar{\mathcal{B}_i} \cap \bar{\mathcal{C}_i} \cap \bar{\mathcal{D}_i}) + \ \Pr(\bar{\mathcal{D}}_{i+1} \mid \mathcal{A}_i \cap \mathcal{B}_i \cap \mathcal{C}_i \cap \mathcal{D}_i)$.
Now if $\mathcal{C}_i \cap \mathcal{D}_i$ occurs then

$$\begin{aligned} |V_i|(1-p)^{|S_i|} &\ge n \left(1 - \frac{2}{(\log n)^2}\right)^{i-1} (1-p)^{|S_1| + |S_2| + \ldots + |S_i|} \\ &\ge n \left(1 - \frac{2}{(\log n)^2}\right)^{i-1} (1-p)^{(1-\alpha)\log n/2p} \\ &= (1 - o(1)) n^{1 + \frac{1-\alpha}{2p}\log(1-p)} \end{aligned}$$

and $|Y| \ge |V_i| - \frac{(\log n)^2}{p} \ge (1 - \frac{1}{\log n)^2})|V_i|$.
Now, since $\mathcal{C}_i, \mathcal{D}_i$ refer to the history of the algorithm prior to the construction of $Y \cap V_{i+1}$ we may again argue as in (9) that

$$\Pr(\bar{\mathcal{D}}_{i+1} \mid \mathcal{A}_i \cap \mathcal{B}_i \cap \mathcal{C}_i \cap \mathcal{D}_i) \le \exp\left\{ -\frac{(1 - o(1)) n^{1 + \frac{1-\alpha}{2p}\log(1-p)}}{2(\log n)^4} \right\} .$$

Thus, from (6), (7), (8), (10) and the above

$$\Pr(\bar{\mathcal{D}}_{i+1}) \leq \Pr(\bar{\mathcal{D}}_i) + o((\log n)^{-1})$$

and so

$$\begin{aligned}\Pr(\bar{\mathcal{D}}_{i+1}) &\leq \Pr(\bar{\mathcal{D}}_1) + o(1) \\ &= o(1).\end{aligned}$$

since $\bar{\mathcal{D}}_1 = \emptyset$.
Thus $\Pr(\bar{\mathcal{D}}_\tau) = o(1)$. Combining this with $\Pr(\mathcal{C}_\tau) = 1 - o(1)$ we see that

$$\Pr(V_\tau = \emptyset) = o(1)$$

and this proves part (b) of the Theorem. □

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