A CURIOUS BINOMIAL IDENTITY

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In this note we shall prove the following curious identity of sums of powers of the partial sum of binomial coefficients.

1. AN IDENTITY

**Theorem.** \( \sum_{l=0}^{n} \left( \sum_{k=0}^{l} \binom{n}{k} \right)^3 = n2^{3n-1} + 2^{3n} - \frac{3n}{4}2^n \binom{2n}{n} \).

**Proof.** Define \( f_n = \sum_{l=0}^{n} \left( \sum_{k=0}^{l} \binom{n}{k} \right)^3 \). It is sufficient to show that

\[
f_{n+1} - 8f_n = 4 \cdot 2^{3n} - 3 \cdot 2^n \binom{2n}{n}
\]

Write \( A_l = \sum_{k=0}^{l} \binom{n}{k} \). Then \( f_n = \sum_{l=0}^{n} A_l^3 \).

\[
f_{n+1} = \sum_{l=0}^{n+1} \left( \sum_{k=0}^{l} \binom{n+1}{k} \right)^3
\]

\[
= 2^{3n+3} + \sum_{l=0}^{n} \left( \sum_{k=0}^{l} \binom{n+1}{k} \right)^3
\]

\[
= 2^{3n+3} + \sum_{l=0}^{n} \left( \sum_{k=0}^{l} \binom{n}{k} + \binom{n}{k-1} \right)^3
\]

\[
= 2^{3n+3} + \sum_{l=0}^{n} \left( 2A_l - \binom{n}{l} \right)^3
\]

\[
f_{n+1} - 8f_n = 2^{3n+3} + \sum_{l=0}^{n} \left( 2A_l - \binom{n}{l} \right)^3 - (2A_l)^3
\]

\[
= 2^{3n+3} - \sum_{l=0}^{n} 12A_l^2 \binom{n}{l} + \sum_{l=0}^{n} 6A_l \binom{n}{l}^2 - \sum_{l=0}^{n} \binom{n}{l}^3
\]

**Observation 1:**

\[
\sum_{l=0}^{n} A_l \binom{n}{l}^2 = \frac{1}{2} \cdot 2^n \binom{2n}{n} + \frac{1}{2} \sum_{l=0}^{n} \binom{n}{l}^3
\]

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Indeed;

\[
\sum_{l=0}^{n} A_l \binom{n}{l}^2 = \sum_{l=0}^{n} A_{n-l} \binom{n}{n-l}^2
= \sum_{l=0}^{n} A_{n-l} \binom{n}{l}^2
\]

and since

\[
A_l + A_{n-l} = 2^n + \binom{n}{l}
\]

we have

\[
\sum_{l=0}^{n} A_l \binom{n}{l}^2 = \frac{1}{2} \sum_{l=0}^{n} \left(2^n + \binom{n}{l}\right) \binom{n}{l}^2
= \frac{1}{2} \sum_{l=0}^{n} 2^n \binom{n}{l}^2 + \frac{1}{2} \sum_{l=0}^{n} \binom{n}{l}^3
= \frac{1}{2} 2^n \binom{2n}{n} + \frac{1}{2} \sum_{l=0}^{n} \binom{n}{l}^3
\]

Observation 2:

\[
\sum_{l=0}^{n} A_l^2 \binom{n}{l} = \frac{2^{3n}}{3} + \frac{1}{2} 2^n \binom{2n}{n} + \frac{1}{6} \sum_{l=0}^{n} \binom{n}{l}^3
\]

Indeed,

\[
2^{3n} = A_n^3 = \sum_{l=0}^{n} A_l^3 - A_{l-1}^3
= \sum_{l=0}^{n} A_l^3 - \left(A_l - \binom{n}{l}\right)^3
= \sum_{l=0}^{n} 3A_l^2 \binom{n}{l} - \sum_{l=0}^{n} 3A_l \binom{n}{l}^2 + \sum_{l=0}^{n} \binom{n}{l}^3
= \sum_{l=0}^{n} 3A_l^2 \binom{n}{l} - \frac{3}{2} 2^n \binom{2n}{n} - \frac{1}{2} \sum_{l=0}^{n} \binom{n}{l}^3
\]

Hence

\[
\sum_{l=0}^{n} A_l^2 \binom{n}{l} = \frac{2^{3n}}{3} + \frac{1}{2} 2^n \binom{2n}{n} + \frac{1}{6} \sum_{l=0}^{n} \binom{n}{l}^3
\]

Putting these together, we indeed find that

\[
f_{n+1} - 8f_n = 4 2^{3n} - 3 \cdot 2^n \binom{2n}{n}
\]

as required. \(\square\)
2. An application

In this section we shall discuss an application of this to order statistics. Observe that the expected value of the maximum of three independent Bernoulli random variables \( B(n, \frac{1}{2}) \) is

\[
\sum_{l=0}^{n} \left( 1 - \left( \sum_{k=0}^{l} 2^{-n} \binom{n}{k} \right)^3 \right) = n - 2^{-3n} f_n
\]

\[
= n + \frac{3}{4} n 2^{-2n} \binom{2n}{n}.
\]

Hence, by the central limit theorem, the expected value \( m_3 \) of the maximum of three independent normal \( \mathcal{N}(0, 1) \) random variables is

\[
m_3 = \lim_{n \to \infty} \frac{\frac{3}{4} n 2^{-2n} \binom{2n}{n}}{\frac{\sqrt{n}}{2}} = \frac{3}{2 \sqrt{\pi}}
\]

subtracting off the mean, dividing by the standard deviation and applying Stirling’s formula for the asymptotics of \( n! \).

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