A CURIOUS BINOMIAL IDENTITY

NEIL J. CALKIN

In this note we shall prove the following curious identity of sums of powers of the partial sum of binomial coefficients.

1. An identity

Theorem . $\sum_{l=0}^{n}\left(\sum_{k=0}^{l}\binom{n}{k}\right)^3 = n2^{3n-1}+2^{3n}-\frac{3n}{4}2^n\binom{2n}{n}$.

Proof. Define $f_n = \sum_{l=0}^{n}\left(\sum_{k=0}^{l}\binom{n}{k}\right)^3$. It is sufficient to show that

$$f_{n+1}-8f_n = 4\ 2^{3n} - 3\ 2^n\binom{2n}{n}$$

Write $A_l = \sum_{k=0}^{l}\binom{n}{k}$. Then $f_n = \sum_{l=0}^{n} A_l^3$.

$$\begin{aligned}
f_{n+1} &= \sum_{l=0}^{n+1}\left(\sum_{k=0}^{l}\binom{n+1}{k}\right)^3\\
&= 2^{3n+3} + \sum_{l=0}^{n}\left(\sum_{k=0}^{l}\binom{n+1}{k}\right)^3\\
&= 2^{3n+3} + \sum_{l=0}^{n}\left(\sum_{k=0}^{l}\binom{n}{k}+\binom{n}{k-1}\right)^3\\
&= 2^{3n+3} + \sum_{l=0}^{n}\left(2A_l-\binom{n}{l}\right)^3\\
f_{n+1}-8f_n &= 2^{3n+3} + \sum_{l=0}^{n}\left(2A_l-\binom{n}{l}\right)^3 - (2A_l)^3\\
&= 2^{3n+3} - \sum_{l=0}^{n} 12A_l^2\binom{n}{l} + \sum_{l=0}^{n} 6A_l\binom{n}{l}^2 - \sum_{l=0}^{n}\binom{n}{l}^3
\end{aligned}$$

Observation 1:

$$\sum_{l=0}^{n} A_l\binom{n}{l}^2 = \frac{1}{2}2^n\binom{2n}{n} + \frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}^3$$

1991 *Mathematics Subject Classification.* 05A10.

Indeed;

$$\begin{aligned}\sum_{l=0}^{n} A_l \binom{n}{l}^2 &= \sum_{l=0}^{n} A_{n-l} \binom{n}{n-l}^2 \\ &= \sum_{l=0}^{n} A_{n-l} \binom{n}{l}^2\end{aligned}$$

and since

$$A_l + A_{n-l} = 2^n + \binom{n}{l}$$

we have

$$\begin{aligned}\sum_{l=0}^{n} A_l \binom{n}{l}^2 &= \frac{1}{2} \sum_{l=0}^{n} \left(2^n + \binom{n}{l}\right) \binom{n}{l}^2 \\ &= \frac{1}{2} \sum_{l=0}^{n} 2^n \binom{n}{l}^2 + \frac{1}{2} \sum_{l=0}^{n} \binom{n}{l}^3 \\ &= \frac{1}{2} 2^n \binom{2n}{n} + \frac{1}{2} \sum_{l=0}^{n} \binom{n}{l}^3\end{aligned}$$

Observation 2:

$$\sum_{l=0}^{n} A_l^2 \binom{n}{l} = \frac{2^{3n}}{3} + \frac{1}{2} 2^n \binom{2n}{n} + \frac{1}{6} \sum_{l=0}^{n} \binom{n}{l}^3$$

Indeed,

$$\begin{aligned}2^{3n} = A_n^3 &= \sum_{l=0}^{n} A_l^3 - A_{l-1}^3 \\ &= \sum_{l=0}^{n} A_l^3 - \left(A_l - \binom{n}{l}\right)^3 \\ &= \sum_{l=0}^{n} 3A_l^2 \binom{n}{l} - \sum_{l=0}^{n} 3A_l \binom{n}{l}^2 + \sum_{l=0}^{n} \binom{n}{l}^3 \\ &= \sum_{l=0}^{n} 3A_l^2 \binom{n}{l} - \frac{3}{2} 2^n \binom{2n}{n} - \frac{1}{2} \sum_{l=0}^{n} \binom{n}{l}^3\end{aligned}$$

Hence

$$\sum_{l=0}^{n} A_l^2 \binom{n}{l} = \frac{2^{3n}}{3} + \frac{1}{2} 2^n \binom{2n}{n} + \frac{1}{6} \sum_{l=0}^{n} \binom{n}{l}^3$$

Putting these together, we indeed find that

$$f_{n+1} - 8f_n = 4\ 2^{3n} - 3\ 2^n \binom{2n}{n}$$

as required. $\square$

2. An application

In this section we shall discuss an application of this to order statistics. Observe that the expected value of the maximum of three independent Bernoulli random variables $B(n, \frac{1}{2})$ is

$$\sum_{l=0}^{n} \left(1 - \left(\sum_{k=0}^{l} 2^{-n}\binom{n}{k}\right)^3\right) = n - 2^{-3n} f_n$$
$$= \frac{n}{2} + \frac{3}{4} n 2^{-2n}\binom{2n}{n}.$$

Hence, by the central limit theorem, the expected value m_3 of the maximum of three independent normal $N(0,1)$ random variables is

$$m_3 = \lim_{n\to\infty} \frac{\frac{3}{4} n 2^{-2n}\binom{2n}{n}}{\frac{\sqrt{n}}{2}} = \frac{3}{2\sqrt{\pi}}$$

subracting off the mean, dividing by the standard deviation and applying Stirling's formula for the asymptotics of $n!$

School of Mathematics, Georgia Institute of Technology, Atlanta, Ga 30332
E-mail address: calkin@math.gatech.edu