Binary partitions of integers and Stern-Brocot-like trees

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Abstract

We study partitions of an integer n in which all parts are powers of 2, no power of 2 appearing more than twice. If $b(n)$ is the number of these, and $B(n) = \sum_{j \le n} b(j)$, then we find (a) that $\gcd(b(n), b(n+1)) = 1$, and that every coprime (i, j) occurs exactly once as a pair of consecutive values (b) the asymptotic behavior of $B(t2^n)$, for fixed $1 < t < 2$, as $n \to \infty$ (c) descriptions of $b(n)$ in terms of the structure of the blocks of 0's and 1's in the binary expansion of n, and many other properties of these functions. This partition function has previously been studied, under different guises, in other areas of discrete mathematics, such as in the theory of the Stern-Brocot tree.

1 Introduction and statement of results

In the September-October 1997 issue of *Quantum*, the following question appeared: "How many ways can 90316 be written as $a + 2b + 4c + 8d + 16e + 32f + \ldots$, where the coefficients can be any of 0, 1, or 2?" In combinatorial language, we are being asked about the number of partitions of an integer n into powers of 2 (*binary* partitions), in which the multiplicity of each part is at most 2. Unrestricted binary partitions have been extensively studied [Andr]. In this paper we study binary partitions with the additional strong restriction that the multiplicities of the parts are all ≤ 2. We let $b(n)$ denote the number of such partitions of n, we write $B(n)$ for the partial sums $b(0) + b(1) + \ldots + b(n)$, we use $\lg x$ for $\log_2 x$, and we write "$[x^n]\{\ldots\}$" for the coefficient of x^n in the expression "$\ldots$".

The functions $b(n)$ and $B(n)$ have a number of remarkable properties. To see some examples of these, here is an initial segment of the generating function of the $b(n)$'s:

$$\begin{aligned}\sum_{n\ge 0} b(n)x^n &= \prod_{j\ge 0}(1 + x^{2^j} + x^{2^{j+1}}) \qquad (1)\\ &= 1 + x + 2x^2 + x^3 + 3x^4 + 2x^5 + 3x^6 + x^7 + 4x^8 + 3x^9 + 5x^{10} + 2x^{11} + 5x^{12} + 3x^{13}\\ &\quad + 4x^{14} + x^{15} + 5x^{16} + 4x^{17} + 7x^{18} + 3x^{19} + 8x^{20} + 5x^{21} + 7x^{22} + 2x^{23} + 7x^{24} + 5x^{25}\\ &\quad + 8x^{26} + 3x^{27} + 7x^{28} + 4x^{29} + 5x^{30} + x^{31} + 6x^{32} + 5x^{33} + 9x^{34} + 4x^{35} + 11x^{36} + 7x^{37}\end{aligned}$$

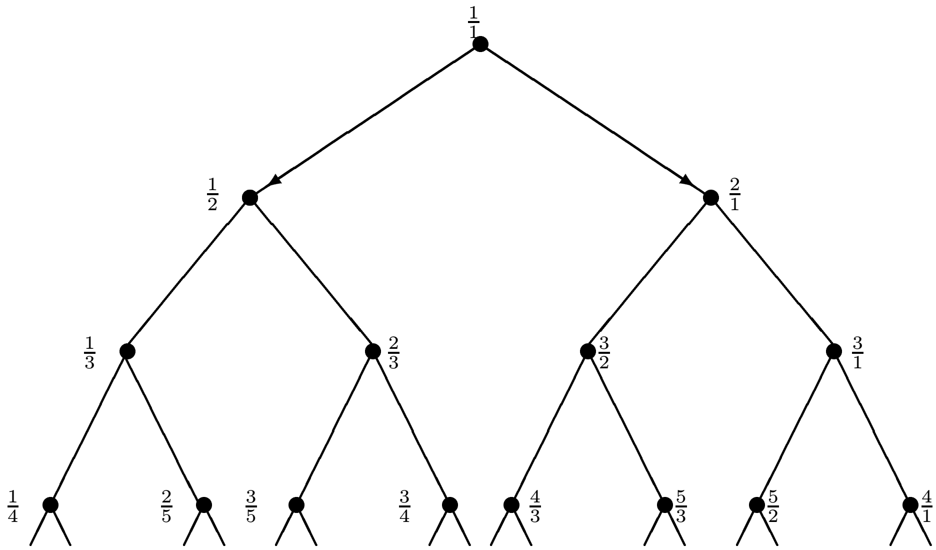

Figure 1: The Stern-Brocot tree

$$+10\,x^{38} + 3\,x^{39} + 11\,x^{40} + 8\,x^{41} + 13\,x^{42} + 5\,x^{43} + 12\,x^{44} + 7\,x^{45} + 9\,x^{46} + 2\,x^{47} + \ldots.$$

- Note that $b(n)$ and $b(n+1)$ seem always to be relatively prime, and that no relatively prime ordered pair (i, j) seems to occur more than once.

- Observe that the consecutive values of $b(n)$ on values of n that are a fixed number of steps prior to a power of 2 appear to be in arithmetic progression, e.g., the values $b(2^j - 4)$ are $\{1, 3, 5, 7, 9, \ldots\}$.

Many beautiful patterns appear in the sequence $\{b(n)\}$, which is seemingly in such a lowly position in the hierarchy of partition problems. However we are scarcely the first to study this function:

- It is sequence M0141 in [SlPl].

- It occurs in the theory of the Stern-Brocot tree [Br, GKP, St]. This is an infinite binary tree, with a simple rule for labeling the children of a given labeled vertex, in which each positive rational number, in reduced form, occurs once and only once as the label of a vertex. It is shown in Fig. 1. Notice that as we move from left to right across any level of the tree, the sequence of numerators of the fractions that appear is our sequence $\{b(n)\}$. The sequence of denominators on a level is the reversed sequence of numerators.

- The sequence $\{b(n)\}$ also occurs in combinatorial game theory, specifically $b(n)$ is the number "of nim-sums corresponding to a given ordinary sum $N, \ldots$" [BCG].

In none of these works has $b(n)$ been recognized in its role as a partition function, as we do here.

Here is a summary of our main results. First, regarding the growth of $B(n)$ (see section 7), we have

Theorem 1 *If $B(n)$ is the total number of partitions of all integers $\leq n$ into powers of 2, with the multiplicity of each part being at most 2, write $y(n) = B(n)/n^{\lg 3}$. For t real, $1 < t < 2$, the limit*

$$\lim_{n\to\infty} y(t2^n) \stackrel{def}{=} f(t)$$

exists and is a (nonvanishing, finite) continuous function of t. Explicitly, if t is a dyadic fraction $t = 1 + 2^{-n_1} + 2^{-n_2} + \ldots + 2^{-n_d}$ between 1 and 2, then we have

$$f(t) = \frac{2B(N_0 - 1) - b(N_0 - 1)}{N_0^{\lg 3}}, \tag{2}$$

in which $N_0 = N_0(t) = 2^{n_d} + 2^{n_d - n_1} + \ldots + 1$.

Concerning the matter of relative primality of consecutive values, we have (see section 2) the following.

Theorem 2 *For all $n = 0, 1, 2, \ldots$ we have $\gcd(b(n), b(n+1)) = 1$. Further, let (i, j) be an ordered pair of relatively prime positive integers. Then there exists a unique $n = n(i, j)$ such that $b(n) = i$ and $b(n+1) = j$.*

Regarding the relationship between $b(n)$ and the Stern-Brocot tree, we give (see section 2) another natural labelling of the infinite binary tree by all reduced rationals, which has the rather startling property that if we define unique walks from the root to a given label in a tree, writing "L" (resp. "R") each time we take a step to the left (resp. to the right), then the following holds.

Theorem 3 *For a given fraction i/j, the L-R words for the two trees are reversals of each other. That is, the path from the root to the fraction i/j in the Stern-Brocot tree and the path from (i, j) to the root in our tree are identical, when regarded as words of L's and R's.*

Our tree can be described as follows.

Theorem 4 *Construct a binary tree T whose vertices are ordered pairs of positive integers, as follows: (a) $(1, 1) \in T$, and (b) the left-child of (i, j) is $(i, i + j)$ and its right-child is $(i + j, j)$. Then every pair (i, j) of relatively prime positive integers occurs once and only once as the label of a vertex in T, and conversely every vertex label is such a pair. As we read the vertex labels from top to bottom and from left to right we see the consecutive values of our partition function $b(n)$, in order, as their numerators (see Fig. 2 below).*

Next, we describe $b(n)$, in various ways, in terms of the structure of the blocks of 0's and 1's in the binary expansion of n.

Definition 1 *If n is a d-bit binary integer, then by an alternating bit set in n, abbreviated a.b.s., we mean a subset of the d bit positions of n with the property that the bits of n that lie in those positions are alternately 1's and 0's, the leftmost (most significant) of them being a 1 and the rightmost (least significant) being a 0.*

Then (see section 4):

Theorem 5 *$b(n)$ is equal to the number of alternating bit sets of n.*

Normally one does not expect to find a close correspondence between a truncated product and a truncation of the series that represents the full product. In this case, though, we have (see section 8) the following.

Theorem 6 *The reversed partial sums of the generating series of $\{b(n)\}$ are given in terms of the partial products of its defining product by*

$$\sum_{j=0}^{2^{n+1}-1} b(j) x^{2^{n+1}-j} = x + x^2 + x^2 \sum_{j=0}^{n-1} \prod_{\ell=0}^{j} (1 + x^{2^\ell} + x^{2^{\ell+1}}) \tag{3}$$

We deduce a number of consequences of this, one of which is the following, which shows that the sequence $\{b(2^n - j)\}_n$ is an arithmetic progression.

Theorem 7 *There exists $\alpha(m)$, defined for $m \geq 2$, such that for all j such that $2^j > m \geq 2$ we have $b(2^j - m) = jb(m-2) - \alpha(m)$. In fact, $\alpha(m)$ is given by*

$$\alpha(m) = (1 + \lceil \lg k \rceil) b(m-2) - [x^{m-2}] \sum_{0 \leq j < \lg(m-2)} \prod_{\ell=0}^{j} (1 + x^{2^\ell} + x^{2^{\ell+1}}). \tag{4}$$

Beyond these theorems we find a number of properties and special values of $b(n)$ and $B(n)$, as well as developing a 2×2 matrix formalism (Theorem 9) with which one can find $b(n)$ in terms of the constant blocks of binary digits of n, and another matrix formalism that expresses $b(n)$ in terms of the inverse of a $k \times k$ triangular matrix, where $2k$ is the number of constant blocks of bits in $(n)_2$.

Here are some of the elementary properties of the sequences $\{b(n)\}$, $\{B(n)\}$, collected for future reference.

If $\mathcal{B}(x)$ is the generating function (1), then clearly $(1 + x + x^2)\mathcal{B}(x^2) = \mathcal{B}(x)$, from which we have

Proposition 1 $\forall n \geq 1$, $b(2n) = b(n) + b(n-1)$, *and* $\forall n \geq 0$, $b(2n+1) = b(n)$. $\square$

Proposition 2 $\forall n \geq 0$ *we have* $B(2n) = 3B(n) - 2b(n)$.

Proof. Add together the equations $1 = 1$, $b(0) = b(1)$, $b(1) + b(0) = b(2)$, $b(1) = b(3)$, $b(2) + b(1) = b(4)$, $\ldots$, $b(n) + b(n-1) = b(2n)$. $\square$

Proposition 3 $\forall n \geq 0$ *we have* $B(2n+1) = 3B(n) - b(n)$.

Proof. Add $b(n) = b(2n+1)$ to the previous Proposition. $\square$

Proposition 4 *We have* $b(k2^m) = b(k) + mb(k-1)$ *and* $b(k2^m - 1) = b(k-1)$.

Proof. If $m = 0$ this is clear. If true for $m - 1$ then

$$b(k2^m) = b(2(k2^{m-1})) = b(k2^{m-1}) + b(k2^{m-1} - 1) = (b_k + (m-1)b(k-1)) + b(k-1) = b_k + mb(k-1),$$

and $b(k2^m - 1) = b(k2^{m-1} - 1) = b(k-1)$ by Prop. 1 and induction. $\square$

2 A Stern-Brocot-like tree

The recurrence in Proposition 1 gives rise to a pretty binary tree of values of $\{b(n)\}$. Suppose, at the vertices of an infinite binary tree we put the ordered pairs $(b(j), b(j+1))$, $(j \geq 0)$ of consecutive values of $b(n)$. Suppose further that the two children of the parent $b(j), b(j+1)$ are $b(2j), b(2j+1)$ and $b(2j+2), b(2j+3)$. We then obtain the tree that is shown in Fig. 2, in which we have displayed the pairs (i, j) at each vertex as fractions i/j.

The rule by which this tree is formed can be stated recursively as follows:

- $(1, 1)$ is a vertex of the tree, and
- The children of vertex (i, j) are $(i, i + j)$ and $(i + j, j)$.

Here are some of the properties of this tree.

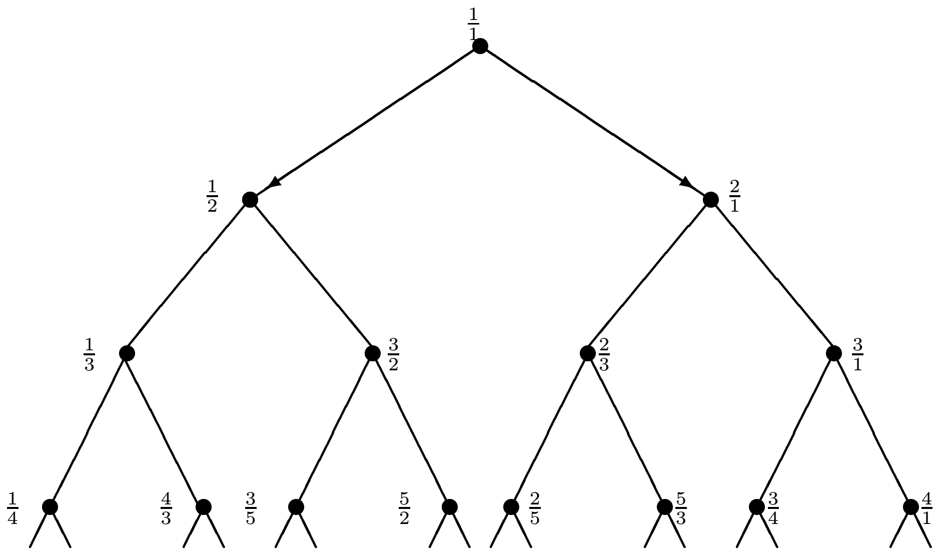

Figure 2: A Stern-Brocot-like tree

Theorem 8 *Every positive rational number r/s in reduced form occurs as the label of one and only one vertex of the tree, and every vertex is labeled by such a reduced fraction.*

Proof. If some fraction m/n does not occur, choose one with minimum n, and among those, one of minimum m. If $m = n$ then $m = n = 1$, which does occur. If $m < n$ then $(m, n - m)$ cannot occur, contradicting the minimality, and similarly if $m > n$ then $(m - n, n)$ cannot occur, which is again a contradiction.

The fraction r/s, when it occurs as a vertex label, does so in lowest terms, i.e., with $\gcd(r, s) = 1$. Indeed, the root $(1, 1)$ does so, and if some vertex label (r, s) has gcd > 1 then its parent, namely $(r, s - r)$, if $s > r$, or $(r - s, s)$, if $r > s$, will also, which contradicts induction down the levels of the tree. □.

The argument above yields an explicit algorithm that finds the unique integer $n = n(i, j)$ such that $b(n - 1) = i$ and $b(n) = j$:

```
If i=j=1 then return 1
         else if i>j then return 1+2n(i-j,j)
                     else return 2n(i,j-i)
```

□

Corollary 1 *Theorem 2 is true.*

Here is an open question. Given a fraction i/j, how can we find an upper estimate for the level of the tree on which i/j lives? This level number is $r = a_1 + a_2 + \ldots a_k$ where the continued fraction expansion of

i/j is

$$\frac{i}{j} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \dots \cfrac{1}{a_k}}}}}$$

The reason is that each step from a level to an adjacent level corresponds to either replacing i/j by $i/(j-i)$ or $(j-i)/i$, with the denominator larger than the numerator. Now, if we take r steps before reaching i/j, then $n < 2^r$: hence if we can bound $a_1 + a_2 + \dots a_k$, we can bound n.

Thus the problem of bounding n is closely related to the problem of minimizing $a_1 + a_2 + \dots a_k$ for all fractions i/j with $1 < i < j$ and $\gcd(i,j) = 1$.

There is a close relationship between this tree of fractions and the Stern-Brocot tree. Fix a fraction m/n. Suppose, in the Stern-Brocot tree, we follow the unique walk from the root to (m,n), writing "L" (resp. "R") each time we take a step to the left (resp. to the right). Then each fraction is made to correspond to a certain finite L-R word.

Likewise, suppose that in our binary tree we do the same thing, and we obtain an L-R word for a certain fraction m/n. Then the somewhat surprising result stated as Theorem 3 holds, namely that these two L-R words are reversals of each other. That is, the path *from* the root *to* a fraction i/j in the Stern-Brocot tree and the path *from* (i,j) *to* the root in our tree are identical, when regarded as words of L's and R's.

The proof is deferred until the next section since it requires the matrix machinery that is developed there.

3 A matrix method

We will now determine an expression for $b(n)$ in terms of products of matrices. First express n in binary as blocks of 1's and 0's. Note that by Prop. 1, any trailing 1's do not change the value of $b(n)$, so w.l.o.g. we can suppose that

$$n = 1^{u_m} 0^{v_m} 1^{u_{m-1}} 0^{v_{m-1}} \dots 1^{u_1} 0^{v_1},$$

where $u_i, v_i \geq 1$, $1 \leq i \leq m$.

Let n' be derived from n by removing the leftmost block of 1's and the leftmost block of 0's, so

$$n' = 1^{u_{m-1}} 0^{v_{m-1}} 1^{u_{m-2}} 0^{v_{m-2}} \dots 1^{u_1} 0^{v_1}.$$

For example, if $n = 54$, with binary expansion 110110, then $n' = 6$, with binary expansion 110. Let $l(n) = \lfloor \lg n \rfloor$ be the number of digits in the binary expansion of n.

Proposition 5 *If n has binary expansion $1^u 0^v$, where $u, v \geq 0$, then $b(n) = uv + 1$.*

Proof. Trivial. □

Now let $\bar{b}(n)$ be the number of representations of n which don't use $2^{l(n)}$, that is, for which the leading bit is shifted to the right. Then, with n' defined as above,

$$b(n) = (u_l v_l + 1) b(n') + u_l \bar{b}(n')$$

and

$$\bar{b}(n) = v_l b(n') + \bar{b}(n')$$

so

$$\begin{pmatrix} b(n) \\ \bar{b}(n) \end{pmatrix} = \begin{pmatrix} u_l v_l + 1 & u_l \\ v_l & 1 \end{pmatrix} \begin{pmatrix} b(n') \\ \bar{b}(n') \end{pmatrix}$$

$$= \begin{pmatrix} u_l v_l + 1 & u_l \\ v_l & 1 \end{pmatrix} \begin{pmatrix} u_{l-1} v_{l-1} + 1 & u_{l-1} \\ v_{l-1} & 1 \end{pmatrix} \cdots \begin{pmatrix} u_1 v_1 + 1 & u_1 \\ v_1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

since if $n = 2^j - 1$, $b(n) = 1$ and $\bar{b}(n) = 0$.

Now,

$$\begin{pmatrix} uv + 1 & u \\ v & 1 \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^u \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^v$$

and

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^u \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Hence, if n has binary expansion $a_k a_{k-1} a_{k-2} \ldots a_0$, then

$$b(n) = (1 \;\; 0) \prod_{i=0}^{k} M(a_{k-i}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where

$$M(x) = \begin{pmatrix} 1 & x \\ 1 - x & 1 \end{pmatrix}.$$

We have thus shown

Theorem 9 *Let* $n = \sum_{i=0}^{k} a_i 2^i$ $(a_i = 0, 1)$ *and define*

$$M(x) = \begin{pmatrix} 1 & x \\ 1 - x & 1 \end{pmatrix}.$$

Then

$$b(n) = (1 \;\; 0) \prod_{i=0}^{k} M(a_{k-i}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

As an immediate consequence we have

Theorem 10 *Let* F_k*,* $k = 0, 1, \ldots$ *be the Fibonacci sequence,* $F_0 = 0, F_1 = 1$*. Then*

$$\max_{n < 2^k} b(n) = F_{k+1}.$$

Proof It is trivial to show by induction that if $n < 2^k$ then $b(n) \leq F_{k+1}$. To show that the bound is attained, compute $b(n)$ for $n = 2(4^k - 1)/3$ and $n = 4(4^k - 1)/3$ using Theorem 9. □

We can now also prove Theorem 3, which states that paths in the Stern-Brocot tree and in our tree are reversals of each other.

First, in the Stern-Brocot tree, we use the matrix representation of steps in that tree ([GKP]). We get from the root to a fraction r/s via a sequence of L's and R's, and this is expressed by the matrix equation

$$\begin{pmatrix} s \\ r \end{pmatrix} = \text{Product}[L's \text{ and } R's] \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where $L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Solving for the root,

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \text{Product}[L^{-1}s \text{ and } R^{-1}s] \begin{pmatrix} s \\ r \end{pmatrix},$$

in which the order of the matrices in the second equation is the reverse of that in the first, and

$$L^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad R^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

But

$$R^{-1} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v-u \end{pmatrix}$$

and

$$L^{-1} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u-v \\ v \end{pmatrix},$$

and these are just the matrices that correspond to downward movements in the CW tree. So the phenomenon essentially amounts to the fact that the inverse of a product of matrices is the product of inverses in reverse order. □

4 A combinatorial interpretation for $b(n)$

In this section we will find a combinatorial interpretation for $b(n)$, namely Theorem 5 above. We prove this in two ways: by the matrix method and via a direct bijection.

4.1 Proof by matrices.

Recall the definition (Definition 1 in section 1) of an alternating bit set. Thus, if $n = (26)_{10} = (11010)_2$, then $\{5, 3, 2, 1\}$ is one of the alternating bit sets of n.

Now, set $M(x) = I + E(x)$, so that

$$E(x) = \begin{pmatrix} 0 & x \\ 1-x & 0 \end{pmatrix}$$

Then

$$\prod_{i=0}^{k} M(a_{k-i}) = \prod_{i=0}^{k} (I + E(a_{k-i})) = \sum_{S \subseteq \{0,1,\ldots k\}} \left(\prod_{s_j \in S} E(a_{k-s_j}) \right).$$

Since $E(1)^2 = E(0)^2 = 1$, this shows that $b(n)$ is the number of alternating bit sets in n, and we have proved Theorem 5. □

4.2 Bijective proof.

Here is another proof of Theorem 5 which exhibits a direct bijection between alternating bit sets in n, on the one hand, and binary partitions of n with multiplicities at most 2, on the other.

Given an a.b.s. σ in n. For every 1 in σ and the 0 that immediately follows it, do the following: suppose that the 1 and the 0 were in positions $i_1 > i_2 \geq 0$ in the bit string of n. Then delete 2^{i_1} from the usual binary representation of n, and replace it by $2^{i_1-1} + 2^{i_1-2} + \ldots + 2^{i_2+1} + 2 \cdot 2^{i_2}$, whose sum is still 2^{i_1}. The result will be a binary partition of n with multiplicities at most 2.

Conversely, suppose we are given such a partition. In the box below we show, in the top line, the bit string of some integer n. In the second line the up arrows mark the 1's and 0's of a designated alternating bit set in n. In the third line there appears a block of the form $11\ldots12$ between each 1 and its following 0 in the designated a.b.s. In the fourth line is the binary partition with multiplicities at most 2 that corresponds to n and the designated a.b.s. It is obtained simply by adding, without carries, the strings in the first and third lines.

To reverse the construction, if the fourth line (the partition) and the first line (the integer n) in the box below are given, we can recover the second line (the a.b.s.) uniquely as follows. Read the fourth line from left to right until a 2 is encountered. Mark a "1" of the a.b.s. there. Continue to the right until encountering a 2 such that the corresponding bit position of n contains a 0. Mark a "0" of the a.b.s. there. Repeat this process to the right of the marked block until the end of the partition is reached. □

1	1	1	1	1	0	0	0	0	1	1	0	1	1	1	1	1	0	0	1	0	1	1	1	1	1	0	0	0	0	0	0	1	1	0	0	1	1	0	0	0
		↑				↑							↑							↑	↑									↑						↑			↑	
		1	1	1	1	2							1	1	1	1	1	1	1	2	1	1	1	1	1	1	1	1	1	2						1	1	1	2	
1	1	2	2	2	1	2	0	0	1	1	0	1	2	2	2	2	1	1	2	2	2	2	2	2	2	1	1	1	1	2	0	1	1	0	0	2	2	1	2	0

An integer n, an alternating bit set in n, and the corresponding partition

4.3 Counting the alternating bit sets of n

We will deduce from Theorem 5 a rather explicit formula for $b(n)$ in terms of the block structure of the bits of n.

Let n be even. Suppose, as we parse the bits of n from left to right (i.e., from most significant to least significant bit), we see a block of x_1 1's followed by a block of y_1 0's, then x_2 1's, and y_2 0's, etc., finishing with a block of x_r 1's and y_r 0's.

Then n has exactly one a.b.s. of size 0. The number of such sets of size 2, i.e., the number of ways of choosing a 1 bit followed by a 0 bit on its right, is $\sum_{j\geq 1} x_j(y_j + y_{j+1} + \ldots)$. Let's write

$$\gamma_j = y_j + y_{j+1} + \ldots \qquad (j = 1, 2, \ldots, r), \tag{5}$$

which is the number of 0's that lie to the right of the jth block of 1's in $(n)_2$. Then the number of a.b.s. of size 2 is $\sum_j x_j\gamma_j$.

The number of a.b.s. of size 4 is easily seen to be

$$\sum_{i<j} x_i(\gamma_i - \gamma_j)x_j\gamma_j,$$

while the number of size 6 is

$$\sum_{i<j<k} x_i(\gamma_i - \gamma_j)x_j(\gamma_j - \gamma_k)x_k\gamma_k,$$

etc. Evidently we are developing the powers of a certain matrix.

Hence define the $r \times r$ matrix

$$\tau_{i,j} = \begin{cases} x_i(\gamma_i - \gamma_j), & \text{if } 1 \leq i < j \leq r; \\ 0, & \text{else,} \end{cases} \tag{6}$$

in which $\gamma_j = 0$ if $j > r$. Further define $\mathbf{e}$ to be the r-vector of all 1's, and $\mathbf{c} = (x_1\gamma_1, x_2\gamma_2, \ldots, x_r\gamma_r)$. Then the number of a.b.s. in n is

$$1 + (\mathbf{e}, \mathbf{c}) + (\mathbf{e}, \tau\mathbf{c}) + (\mathbf{e}, \tau^2\mathbf{c}) + \ldots = 1 + (\mathbf{e}, (I - \tau)^{-1}\mathbf{c}).$$

Since $\mathbf{z} = (I - \tau)^{-1}\mathbf{c}$ is just the solution vector of the system of linear equations $\mathbf{z} = \tau\mathbf{z} + \mathbf{c}$, we can summarize the discussion above as follows.

Theorem 11 *Let $n > 0$ be even, and suppose that its binary representation contains r blocks of 1's, of lengths $x_1, \ldots, x_r$, and r blocks of 0's, of lengths $y_1, \ldots, y_r$. Define the γ_j's by (5), let $\mathbf{c} = (x_1\gamma_1, \ldots, x_r\gamma_r)$, and define the $r \times r$ matrix τ by (6). Then the number $b(n)$ of representations of n as a sum of powers of 2, each used at most twice, is one more than the sum of the entries of the solution vector z of the equations $\mathbf{z} = \tau\mathbf{z} + \mathbf{c}$.*

Of course if n is odd, then by Prop. 1 we can chop off the terminal block of least significant 1's, and then apply the theorem to the even number that remains.

Example. Consider the case $r = 2$, so that $(n)_2 = 1^{x_1}0^{y_1}1^{x_2}0^{y_2}$. Then τ is the 2×2 matrix whose $(1, 2)$ entry is x_1y_1 and whose other entries vanish, and $\mathbf{c} = (x_1(y_1 + y_2), x_2y_2)$. Then from Theorem 11, we find easily that

$$b(n) = 1 + x_1(y_1 + y_2) + x_2y_2(1 + x_1y_1).$$

For instance,

$$b(34, 326, 446, 048) = b(1^{10}0^71^{13}0^5) = 4736.$$

We can also think of $b(n)$ as the number of paths in a certain directed bipartite graph $G = G(n)$. The "upper" vertices of G correspond to blocks of 1's in $(n)_2$, the "lower" vertices to blocks of 0's. Each vertex has a weight equal to the length of the corresponding block. Each upper vertex is connected by a directed edge to every lower vertex that corresponds to a block of 0's that lies to its right in $(n)_2$. Conversely, each lower vertex is connected by a directed edge to every upper vertex that corresponds to a block of 1's that lies to its right in $(n)_2$. A UL path in G is a path that starts at an upper vertex and ends at a lower vertex, and its weight is the product of the weights of the vertices that are on it. The number of a.b.s.'s in n is the total weight of all UL paths in $G(n)$. In Fig. 1 below we show $G(n)$, for $n = (249484)_{10} = (111100111010001100)_2$, with the weight of each vertex shown in the parenthesis next to its name.

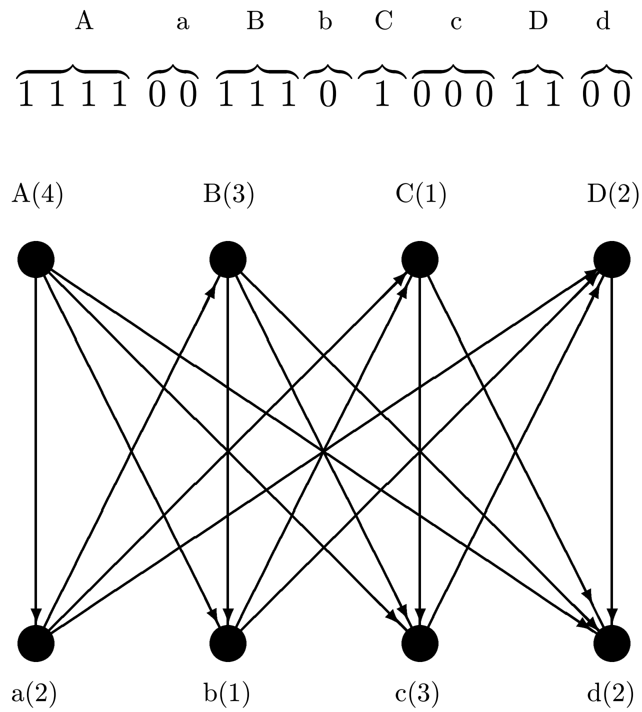

Fig. 1: The graph $G(n)$

5 Evaluating $B(n)$

We can also use Theorem 9 to determine the behaviour of $B(n)$. We begin with the following:

Corollary 2

$$B(2^k - 1) = \frac{3^k + 1}{2}$$

Proof. $B(2^k - 1)$ is the sum of $b(j)$ over all binary numbers with at most k bits: hence it is the sum of the products in the theorem, with every combination of a_i occuring: thus

$$B(2^k - 1) = (1 \;\; 0) \prod_{i=0}^{k-1} (M(0) + M(1))^k \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$= (1 \;\; 0) \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^k \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

which implies $B(2^k-1)=(3^k+1)/2$, as required. $\square$

Write

$$A=\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \tag{7}$$

and

$$\mathbf{e}=\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then we have

Theorem 12 *Let n have binary expansion $a_0a_1a_2\dots a_k$. Then*

$$B(n)=\mathbf{e}^T\sum_{i=0}^{k} a_i\left(\prod_{j=0}^{i-1} M(a_j)\right)M(0)A^{k-i}\mathbf{e}+b(n)$$

Proof: Write the integers up to n as $k+1$-bit binary strings (the standard binary representation with leading 0's). Group the integers by the length of their common prefix with n: observe that there are exactly 2^{k-l} numbers with exactly l bits of common prefix if the next bit of n is a 1, and 0 if the next bit of n is a 1.

For example, if $n=21=(10101)_2$ then the numbers with common prefix exactly 10 are 10000, 10001, 10010, 10011, whereas there are no numbers less than n with common prefix exactly 101. Adding the values of $b(j)$ in each group gives the theorem. $\square$

We will also require bounds on the entries of the product $\prod M(a_j)$.

Proposition 6 *If*

$$N=\prod_{i=1}^{k} M(x_i)=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then $a,b\le F_{k+1}$ and $c,d<F_{k+2}$.

Proof. $a\le F_{k+1}$ follows from Theorem 10. Now, if we consider

$$\bar{N}=\prod_{i=1}^{k} M(1-x_i)$$

we obtain the matrix

$$\begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

which is a matrix of the same form as N, so $d\le F_{k+1}$ too. (In fact, if $x_0=0$, then $a\le F_k, d\le F_{k+1}$, and if $x_0=1$, then $a\le F_{k+1}, b\le F_k$).

Now consider the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} N=\begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \bar{N}=\begin{pmatrix} b+d & c+a \\ b & a \end{pmatrix}.$$

Since these are products of $k+1$ M-matrices, we have $a+c\le F_{k+2}$ and $b+d\le F_{k+2}$: since $a\ge 1$ and $b\ge 1$, we have $c,d<F_{k+2}$, as required. $\square$

6 The behaviour of $B(n)$ for large n

We have seen that $B(2^k - 1) = (3^k + 1)/2$. Since $B(n)$ is an increasing function of n, this implies that $B(n) = \theta(n^{\lg 3})$, that is, that $B(n)/n^{\lg 3}$ is bounded above and below. We shall see that the behaviour of $B(n)/n^{\lg 3}$ for large n is determined by $n/2^k$, where $k = \lfloor \lg n \rfloor$.

Let $t \in [1, 2)$. For each $k \geq 0$, let $n_k = \lfloor t2^k \rfloor$: we shall show that

$$\lim_{k\to\infty} \frac{B(n_k)}{3^k}$$

exists, and is a continuous function of t.

Theorem 13 *For every $t \in [1, 2)$, let t have binary expansion $a_0.a_1a_2a_3\ldots$. Then*

$$\lim_{k\to\infty} \frac{B(n_k)}{3^k} = \mathbf{e}^T \sum_{i=0}^{\infty} \frac{a_i}{3^i} \left(\prod_{j=0}^{i-1} M(a_j) \right) \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Proof. n_k has the binary expansion $a_0a_1a_2\ldots a_k$. Recall that

$$B(n_k) = \mathbf{e}^T \sum_{i=0}^{k} a_i \left(\prod_{j=0}^{i-1} M(a_j) \right) M(0)A^{k-i}\mathbf{e} + b(n_k),$$

where A is given by (7). Since $b(n_k) \leq F_{k+2}$, $b(n_k)/3^k \to 0$ as $k \to \infty$, so it suffices to show that

$$\frac{B(n_k - 1)}{3^k} = \mathbf{e}^T \sum_{i=0}^{k} a_i \left(\prod_{j=0}^{i-1} M(a_j) \right) M(0)A^{k-i}\mathbf{e}/3^k$$

tends to a limit as $k \to \infty$. Now,

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

and hence

$$A^m = \frac{1}{2} \begin{pmatrix} 3^m + 1 & 3^m - 1 \\ 3^m - 1 & 3^m + 1 \end{pmatrix}.$$

Thus

$$\begin{aligned} M(0)A^{k-i}\mathbf{e}/3^k &= \frac{1}{3^k} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3^{k-i} + 1 & 3^{k-i} - 1 \\ 3^{k-i} - 1 & 3^{k-i} + 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{3^i} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{3^k} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

Thus

$$\begin{aligned} \frac{B(n_k - 1)}{3^k} &= \mathbf{e}^T \sum_{i=0}^{k} a_i \left(\prod_{j=0}^{i-1} M(a_j) \right) \left(\frac{1}{3^i} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{3^k} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \\ &= \mathbf{e}^T \sum_{i=0}^{k} \frac{a_i}{3^i} \left(\prod_{j=0}^{i-1} M(a_j) \right) \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{3^k} \sum_{i=0}^{k} a_i b(n_i). \end{aligned}$$

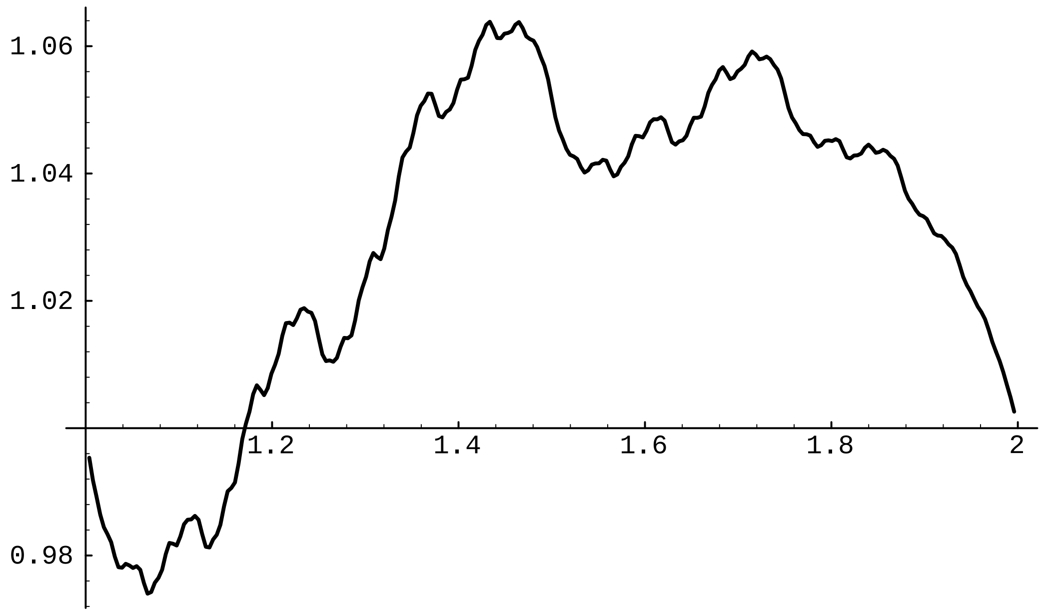

Fig. 2: The limiting function $f(t)$

Now, since $b(n_i) \leq F_{i+2}$, the final term tends to 0 as $k \to 0$. Moreover, since the entries of $\prod_{j=0}^{i-1} M(a_j)$ are bounded above by F_{i+2}, the series

$$\sum_{i=0}^{\infty} \frac{a_i}{3^i} \left(\prod_{j=0}^{i-1} M(a_j) \right)$$

converges, and hence

$$\lim_{k\to\infty} \frac{B(n_k)}{3^k} = \mathbf{e}^T \sum_{i=0}^{\infty} \frac{a_i}{3^i} \left(\prod_{j=0}^{i-1} M(a_j) \right) \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

as claimed. $\Box$

Now define

$$\psi(t) = \mathbf{e}^T \sum_{i=0}^{\infty} \frac{a_i}{3^i} \left(\prod_{j=0}^{i-1} M(a_j) \right) \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and

$$f(t) = \frac{1}{t^{\lg 3}} \psi(t).$$

A graph of $f(t)$ for $1 < t < 2$ is shown in Fig. 2 below.

Clearly, $\psi(t)$ is an increasing function of t on $[1, 2)$: that it is continuous follows from Proposition 6. Thus $f(t)$ is continuous in $(1, 2)$, and it is a straightforward exercise to show that $f(1) = f(2)$. Then we have

Corollary 3 *For each $t \in [1,2)$, let $n_k = \lfloor t2^k \rfloor$. Then*

$$\lim_{k\to\infty} \frac{B(n_k)}{{n_k}^{\lg 3}} = f(t).$$

Proof. Indeed,

$$\lim_{k\to\infty} \frac{3^k}{3^{\lg n_k}} = \frac{3^k}{3^{k+\lg t}} = \frac{1}{3^{\lg t}}. \qquad \square$$

Hence the limit of $B(n)/n^{\lg 3}$, as $n \to \infty$ along a sequence $t2^n$, for a fixed $t \in [1,2]$, is $f(t)$.

This result might be compared with one of the crown jewels of the subject of unrestricted binary partitions, namely the asymptotic formula that was found by de Bruijn [deBr] (see also [Penn]). He showed that the number of all binary partitions of n is of the form

$$(\log n)^{\alpha} n^{\beta} \exp\left(\gamma \log \frac{n}{\log n}\right)^2 \phi\left(\frac{\log n - \log\log n}{\log 2}\right)(1+o(1)) \tag{8}$$

where α, β, γ are certain constants, and ϕ is a periodic function of period 1, which de Bruijn found quite explicitly. Indeed this result of de Bruijn is perhaps the earliest example of an asymptotic formula in the theory of partitions that contains a periodic term.

7 The behavior of $B(k2^n)$ for large n

Now let

$$y_n = \frac{B(n)}{\frac{1}{2}n^{\lg 3}}.$$

Then

$$y_{2n} = \frac{B(2n)}{\frac{1}{2}3n^{\lg 3}} = \frac{3B(n) - 2b(n)}{\frac{1}{2}3n^{\lg 3}} = y_n - \frac{4b(n)}{3n^{\lg 3}}.$$

Thus, for all $k \geq 1$ we have

$$y_{2^k n} = y_n - \frac{4}{n^{\lg 3}} \sum_{j=0}^{k-1} \frac{b(2^j n)}{3^{j+1}} = y_n - \frac{4}{n^{\lg 3}} \sum_{j=0}^{k-1} \frac{b(n) + jb(n-1)}{3^{j+1}}. \tag{9}$$

We remark in passing that if we cancel the factor of $n^{\lg 3}$ in the above, the result is a pretty formula for the $B(n)$'s, viz.,

$$B(n2^k) = 3^k B(n) - b(n)(3^k - 1) + \frac{1}{2}b(n-1)(2k + 1 - 3^k), \tag{10}$$

which gives, for example, the values $B(2^k) = k + (3^k + 3)/2$.

Now let t be a dyadic fraction between 1 and 2, say $t = 1 + 2^{-n_1} + \ldots + 2^{-n_d}$. Put $N_0 = 2^{n_d} + 2^{n_d - n_1} + \ldots + 1$. Then by (9),

$$y_{t2^n} = y_{2^{n-n_d}N_0} = y_{N_0} - \frac{4}{N_0^{\lg 3}} \sum_{j=0}^{n-n_d-1} \frac{(b(N_0) + jb(N_0 - 1))}{3^{j+1}},$$

and taking the limit we find that

$$\begin{aligned}\lim_{n\to\infty} y_{t2^n} &= y_{N_0} - \frac{4}{N_0^{\lg 3}} \sum_{j=0}^{\infty} \frac{(b(N_0) + jb(N_0 - 1))}{3^{j+1}} \\ &= y_{N_0} - \frac{4}{N_0^{\lg 3}} \left(\frac{1}{2} b(N_0) + \frac{1}{4} b(N_0 - 1) \right) \\ &= \frac{2B(N_0 - 1) - b(N_0 - 1)}{N_0^{\lg 3}}\end{aligned}$$

8 Generating polynomials

We will now prove Theorem 6. Consider the product $\prod_{j=0}^{N}(1 + x^{2^j} + x^{2^{j+1}})$. The factors that follow these in the full infinite product can correct only the terms of the series that involve $x^{2^{N+1}}$ and higher powers of x. Hence we surely have

$$\prod_{j=0}^{N}(1 + x^{2^j} + x^{2^{j+1}}) = \sum_{j=0}^{2^{N+1}-1} b(j)x^j + \sum_{j=2^{N+1}}^{2^{N+2}-1} b'(j)x^j,$$

where the quantities $b'(j)$ are to be determined. Now multiply by one more factor, namely by $1 + x^{2^{N+1}} + x^{2^{N+2}}$. Note that after doing that, the coefficients of the powers $0, 1, 2, 3, \ldots, 2^{N+2} - 1$ of x will not change any more if we were to multiply by even more factors in the product, i.e., they will have reached their final values. Hence it must be that

$$b'(j + 2^{N+1}) + b(j) = b(j + 2^{N+1}) \qquad (j = 0, 1, \ldots, 2^{N+1} - 1).$$

Thus we have

$$\prod_{j=0}^{N}(1 + x^{2^j} + x^{2^{j+1}}) = \sum_{j=0}^{2^{N+2}-1} b(j)x^j - x^{2^{N+1}} \sum_{j=0}^{2^{N+1}-1} b(j)x^j,$$

which we abbreviate as $p_N = u_{N+1} - x^{2^{N+1}} u_N$. If we regard this as a first order linear recurrence relation in the u_N's, and "solve it" in the usual way, we get (3). □

Here are some consequences of (3).

1. $\sum_{j=0}^{2^{n+1}-1} b(j) = (3^{n+1} + 1)/2$. (Put $x = 1$ in (3).)

2. $\sum_{j=0}^{2^{n+1}-1} (-1)^j b(j) = (3^n - 1)/2$. (Put $x = -1$ in (3).)

3. $\sum_{j=0}^{2^{n+1}-1} b(j) i^{-j} = -1 - i(3^{n-1} - 1)/2$ $(n \geq 1)$. (Put $x = i$ in (3).)

One more consequence of (3) is Theorem 7, which we now prove. In the sum on the left of (3), put $j := 2^{n+1} - j$ and then replace $b(2^{n+1} - j)$ in that sum by $(n+1)b(j-2) - \alpha(m)$, to see if the equation can then be satisfied by some sequence α that depends only on m. Upon comparing the coefficients of x^{k+2} on both sides of the resulting equation we find that

$$\alpha(k+2) = (n+1)b(k) - [x^k]\sum_{j=0}^{n-1}\prod_{\ell=0}^{j}(1+x^{2^\ell}+x^{2^{\ell+1}}),$$

and we must show that the right side of this equation is independent of n. However, in the sum on the right, all terms for which $j \geq \lceil \lg k \rceil$ have the same coefficient of x^k, namely $b(k)$. Hence we have

$$\begin{aligned}\alpha(k+2) &= (n+1)b(k) - [x^k]\left(\sum_{j<\lg k} + \sum_{\lg k \leq j \leq n-1}\right)\prod_{\ell=0}^{j}(1+x^{2^\ell}+x^{2^{\ell+1}}) \\ &= (n+1)b(k) - [x^k]\sum_{j<\lg k}\prod_{\ell=0}^{j}(1+x^{2^\ell}+x^{2^{\ell+1}}) - (n-1-\lceil \lg k\rceil + 1)b(k)\end{aligned}$$

in which the n's miraculously cancel, and which then simplifies to the result stated in Theorem 7. □

The sequence $\{\alpha(m)\}_2^\infty$ begins as $\{0, 1, 3, 2, 7, 5, 8, 3, 13, 10, 17, 7, 18, 11, 15, 4, 21, 17, 30, \ldots\}$.

9 Values of $b(n)$ in Arithmetic Progression

Motivated by Theorem 7, we consider the values of $b(2^j + m)$: it is clear from computing examples that they lie in arithmetic progression for $j > \lg m$. We give a proof of this using the matrix formulation from Section 3.

Theorem 14 *Let $j > l = \lg m$: then*

$$b(2^j + m) = (j-l+1)b(m) + \bar{b}(m)$$

and

$$b(2^j - m) = b(2^{l+1} - m) + (j-l-1)\bar{b}(2^{l+1} - m)$$

Proof: Let $l = \lg m$ and let m have binary expansion σ: then if $j > l$, the binary expansion of $2^j + m$ is $10^{j-l}\sigma$.

Hence the matrix product becomes

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{j-l}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (j-l+1)a + c & (j-l+1)b + d \\ (j-l)a + c & (j-l)b + d \end{pmatrix}$$

in which $a = b(m)$, and $c = \bar{b}(m)$. Hence $b(2^j + m) = (j-l+1)b(m) + \bar{b}(m)$.

Similarly, $2^j - m = 2^j - 2^{l+1} + 2^{l+1} - m$: writing $m' = 2^{l+1} - m$, and $\sigma(m')$ for the binary expansion of m', $2^j - m$ has binary expansion $1^{j-l-1}\sigma(m')$. The result then follows in the same fashion. □

Similarly, if k and m are integers, we can compute $b(2^j k + m)$ for $j \geq \lg m$ in terms of the entries of the matrix products for k and for m.

An alternative method of proof is to use alternating bit strings.

10 Open Problems

We conclude by listing some open problems. Define $b_{k,l}(n)$ to be the number of partitions of n into powers of l, each part appearing at most k times, so that $b(n) = b_{2,2}(n)$.

It is an easy exercise to show that

$$b_{3,2}(n) = \lfloor \frac{n}{2} \rfloor + 1$$

and more generally that

$$b_{l^2-1,l} = \lfloor \frac{n}{l} \rfloor + 1.$$

1. Is $(l^2 - 1, l)$ the first time that we get smooth growth in $b_{k,l}(n)$ (that is if $l \leq k < l^2 - 1$, then growth is not smooth)?

2. Is $k = l^2 - 1$ the first time we get linear maximum growth?

3. Is $k = l^2 - 1$ the first time we get linear minimum growth?

4. Is $k = l^2 - 1$ the first time we get (non-constant) monotone growth?

5. Is the sequence $b_{k,l}(n)$ ever log-convex?

6. What can be said about consecutive digit sets? (Instead of having the number of parts being $0, 1, \ldots, k$, have $0, t, t+1, t+2, \ldots t+k-1$.)

7. What can be said about general digit sets? This question is related to questions about wavelets and tilings.

8. For which pairs (k, l) is $b_{k,l}(n)$ monotonic in n? It is *not* enough that $k \geq l^2 - 1$: for example, $b_{4,2}$ is not monotonic!

9. Theorem 10 implies that the "maximum order" of $b(n)$ is $n^{\lg \phi}$, where $\phi = (1 + \sqrt{5})/2$. In more detail, that theorem implies that

$$0.958854\ldots = \frac{\phi}{\sqrt{5}} \left(\frac{3}{2}\right)^{\lg \phi} \leq \limsup_{n\to\infty} \frac{b(n)}{n^{\lg \phi}} \leq \frac{1+\phi}{\sqrt{5}} = 1.170820\ldots.$$

   In [BCG] the upper bound 1.25 is derived. But what is the exact value of this limsup?

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