Analytic Number Theory

In 1737, Euler re-proved a classic theorem, that there are infinitely many primes, in a new and beautiful way. He observed that since the function $\zeta(s)$ has both a summation and a product form,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\text{prime}} (1 - \frac{1}{p^s})^{-1}$$

an identity valid for $s > 1$, and since the sum diverges as s approaches 1 from above, the same must be true of the product. This immediately implies that there must be infinitely many primes. This proof is perhaps the first hint of a connection between analysis (the convergence of a series) and algebra/combinatorics (the number of primes).

About 60 years later, Gauss conjectured that $\pi(x)$, the number of primes less than x, is asymptotic to the function $Li(x)$, where

$$Li(x) = \int_2^x \frac{1}{\log x}\, \mathrm{d}x.$$

In 1837, Dirichlet used L-functions, generalizations of the zeta function, to prove that every arithmetic progression $a + kb,\ k = 0, 1, 2, 3, \ldots$ with $\gcd(a, b) = 1$ contains infinitely many primes. In 1859, Riemann demonstrated even deeper connections between the analytic behaviour of the zeta function and the behaviour of $\pi(x)$. He showed that the zeta function can be analytically continued to the whole complex plane, with a single simple pole at $s = 1$, and zeros (the "trivial zeros") at negative even integers. He gave an expression for $\pi(x)$ involving a sum over all the other zeros: this implied a remarkable result: that all of the non-trivial zeros have real part $\frac{1}{2}$ if and only if

$$|Li(x) - \pi(x)| < cx^{1/2} \log(x).$$

Riemann conjectured that the zeros did all lie on the line $\frac{1}{2} + it$.

Following ideas of Riemann, Hadamard and de la Vallée Poussin independently proved Gauss' conjecture, which became the Prime Number Theorem.

Riemann's conjecture was largely ignored until 1900, when Hilbert listed it among his 23 problems at the International Congress of Mathematicians in Paris: this led to some progress, von Mangoldt proving the validity of some of Riemann's claims, Hardy showing that there are infinitely many zeros on the line $\frac{1}{2} + it$, etc, but despite efforts of some of the twentieth century's best mathematicians, the problem remains open. In 2000, the Clay Mathematics Institute announced that the Riemann Hypothesis was one of its seven questions each of which would carry a million dollar prize. This summer, three books about the Riemann Hypothesis were published for the non-mathematical audience (each of them entertaining, some containing more mathematics than others!)

In this course, we will learn some of the techniques of analytic number theory, especially summation by parts and Euler-Maclaurin summation, together with some tools from complex analysis (analytic continuation, functional equations for zeta functions), and see how analysis and number theory intertwine. We will understand the Riemann Hypothesis, although we will not attempt a proof.

We will also apply these techniques to questions such as

- If two large numbers are chosen at random, what is the probability that they have a non-trivial common factor?
- How many prime factors does a "typical" number have?
- How does the function $\phi(n)$ behave? How small can $\phi(n)/n$ be?

The texts we will use are *The Riemann zeta function* by Edwards, published by Dover (and hence cheap) and *The prime numbers and their distribution*, by Mendes-France and Tenenbaum, published by the AMS (and hence also cheap). Currently amazon.com is bundling these two books together for \$27.47. We will also refer on occasion to *Introduction to analytic and probabilistic number theory* by Tenenbaum, published by Cambridge University Press, which is unfortunately only available in hardback, and is expensive: students are not expected to purchase this (admittedly wonderful) book.