Exercise H3.1. Consider Model II for sample size $n = 1$ and prior distribution $\mathcal{L}(\vartheta) = N(\mu, \tau^2)$ where μ is general (possibly nonzero). Recall that Proposition 3.2 concerns the case $\mu = 0$.

(i) Find the posterior distribution $\mathcal{L}(\vartheta|X = x)$.

(ii) Show that the family $\{N(\mu, \tau^2),\ \mu \in \mathbb{R}, \tau^2 > 0\}$ is a conjugate family of prior distributions.

(iii) Find the limits of $\mathcal{L}(\vartheta|X = x)$ for $\tau^2 \to \infty$ and $\tau^2 \to 0$ (assuming σ^2 fixed).

Solution: (i)since $X = \theta + \xi$, where $\mathcal{L}(\xi) = N(0, \sigma^2)$, and $\mathcal{L}(\theta) = N(\mu, \tau^2)$, we have the joint density of X and ξ is

$$p(x, \theta) = \frac{1}{2\pi\sigma\tau} e^{\left[-\frac{(x-\theta)^2}{2\sigma^2} - \frac{(\theta-\mu)^2}{2\tau^2}\right]} \text{ (trivial, please see notes),}$$

and,

$$p(\theta|x) = \frac{p(x, \theta)}{P_X(x)}$$

$$= \frac{\frac{1}{2\pi\sigma\tau} e^{\left[-\frac{1}{2\frac{\sigma^2\tau^2}{\sigma^2+\tau^2}}\left(\theta^2 - 2\frac{x\tau^2+\mu\sigma^2}{\tau^2+\sigma^2}\theta + \frac{x^2\tau^2+\mu^2\sigma^2}{\tau^2+\sigma^2}\right)\right]}}{P_X(x)}$$

$$= C(x, \mu, \tau, \sigma) e^{\left[-\frac{1}{2\frac{\sigma^2\tau^2}{\sigma^2+\tau^2}}\left(\theta - \frac{x\tau^2+\mu\sigma^2}{\tau^2+\sigma^2}\right)^2\right]}.$$

Note that $\int p(\theta|x)\, d\theta = 1$, then we have

$$C(x, \mu, \tau, \sigma) = \frac{1}{(2\pi)^{1/2} (\frac{\sigma^2\tau^2}{\sigma^2+\tau^2})^{1/2}}.$$

This implies $\mathcal{L}(\theta|X = x) = N\left(\frac{x\tau^2+\mu\sigma^2}{\tau^2+\sigma^2}, \frac{\sigma^2\tau^2}{\sigma^2+\tau^2}\right)$.

(ii) Obviously, it is right from (i).

(iii) Since $\lim_{\tau\to\infty} p(\theta|x) = \frac{1}{(2\pi)^{1/2}\sigma} e^{-\frac{1}{2}(x-\theta)^2}$, we have

$$\lim_{\tau\to\infty} \mathcal{L}(\theta|X=x) = N\left(x,\sigma^2\right).$$

And $\lim_{\tau\to 0} p(\theta|x) = 0, \quad x \neq \mu,$

$$\infty, \quad x = \mu\ ,$$

we have

$\lim_{\tau\to 0} \mathcal{L}(\theta|X=x) = \delta_\mu$ (why? easy.), where δ_μ is a measure with probability 1 at μ.

Exercise H3.2 Suppose the data X in a statistical have model take values in a countable set $\mathcal{X}$, i.e.X has a discrete law. Let the class of probability functions be

$$p_\vartheta(x),\ x \in \mathcal{X},\ \vartheta \in \Theta$$

where Θ is an arbitrary parameter space. Suppose T is a statistic with values in a set $\mathcal{T}$.

a) Suppose that the probability function can be represented as

(1) $$p_\vartheta(x) = q_\vartheta(T(x))h(x),$$

where q_ϑ, $\vartheta \in \Theta$ is a class of functions on $\mathcal{T}$ and the function h does not depend on ϑ. Show that T is sufficient. (**Comment:** (1) is called the **Neyman criterion** for sufficency).

b). Show that if T is sufficient then there are h and q_ϑ, $\vartheta \in \Theta$ as above such that (1) holds (i.e. the Neyman criterion is also necessary for sufficency of T).

c) Convince yourself that all problems in homework 1can be solved via the Neyman criterion (no written answer necessary).

Solution: a) we need to show that

$$\frac{P_\theta(X=x)}{P_\theta(T(X)=T(x))}$$

is independent of θ.

We know that $P_\theta(T(X)=T(x)) = \sum_{\{y|T(y)=T(x)\}} P_\theta(X=y)$

$$= \sum_{\{y|T(y)=T(x)\}} q_\theta(T(y))\,h(y)$$

$$= q_\theta(T(x)) \sum_{\{y|T(y)=T(x)\}} h(y)\ .$$

Thus $\frac{P_\theta(X=x)}{P_\theta(T(X)=T(x))} = \frac{h(x)}{\sum_{\{y|T(y)=T(x)\}} h(y)}$, which is independent of θ.

b) if $T(X)$ is sufficient,

$$\frac{P_\theta(X=x)}{P_\theta\left(T(X)=T(x)\right)}$$

is independent of θ.

Define $h(x) = \dfrac{P_\theta(X=x)}{P_\theta\left(T(X)=T(x)\right)}$, which is independent of θ, then

$$P_\theta(X=x) = P_\theta(T(X)=T(x))\,h(x)\,, \text{ where } P_\theta(T(X)=T(x)) \text{ is independent of } \theta.$$

Exercise H3.3 Let $X_1, \dots, X_n$ be independent and identically distributed random variables taking values in $\mathbb{Z}_+$ (the set of nonnegative integers). Let the class of probability functions for X_1 be

$$p_\vartheta(x),\ x \in \mathcal{X},\ \vartheta \in \Theta$$

where Θ is an arbitrary parameter space. For any vector $x \in \mathbb{R}^n$, let

$$T(x) = (x_{[1]}, \dots, x_{[n]})$$

the vector of ordered components, i.e. the unique vector $x_{[1]} \leq \dots \leq x_{[n]}$ which is a permutation of $x_1, \dots, x_n$. Show that, when X is observed, the **order statistic**

$$T(X) = (X_{[1]}, \dots, X_{[n]})$$

is sufficient for ϑ. (**Hint:** Neyman criterion).

Solution: we know $P_\theta(T(X)=T(x)) = \sum_{\{y|T(y)=T(x)\}} P_\theta(X=y)$,

where, in fact, y is a permutation of x. Note that the number $n(x)$of $y's$ is independent of θ, and $P_\theta(X=y_1) = P_\theta(X=y_2)$ for any y_1, y_2 satisfying $T(y_1)=T(y_2)$.

Thus $P_\theta(T(X)=T(x)) = n(x)\,P_\theta(X=x)\,,$

i.e.,

$$P_\theta(X=x) = P_\theta(T(X)=T(x))\,\frac{1}{n(x)}.$$

Since $P_\theta(T(X)=T(x))$ is a function of $T(x)$, $T(X)$ is sufficient from Neyman criterion.