Exercise H4.1. Consider the Gaussian location model with restricted parameter space $\Theta = [-K, K]$, where $K > 0$, sample size $n = 1$ and $\sigma^2 = 1$.

(i) Find the minimax linear estimator T_{LM}.

(ii) Show that T_{LM} is strictly better than the sample mean $\bar{X}_n = X$, everywhere on $\Theta = [-K, K]$ (this implies that X is not admissible).

(iii) Show that T_{LM} is Bayesian in the unrestricted model $\Theta = \mathbb{R}$ for a certain prior distribution $N(0, \tau^2)$, and find the τ^2.

Solution:(i)let's assume the linear estimator $T_{LM} = aX + b$ with $a, b \in R$, then

$$\begin{aligned} E_\theta \left(T\left(x\right) - \theta\right)^2 &= E_\theta \left(aX + b - \theta\right)^2 \\ &= E_\theta (a(X - \theta) + b + (a-1)\theta)^2 \\ &= a^2 E_\theta (X-\theta)^2 + 2a E_\theta \left(X - \theta\right)\left(b + (a-1)\right) + \left(b + (a-1)\,\theta\right)^2 \\ &= a^2 + \left(b + (a-1)\,\theta\right)^2 . \end{aligned}$$

Note that $\sup\limits_\theta E_\theta \left(T\left(x\right) - \theta\right)^2 = a^2 + \left(b + (a-1)\,K\right)^2, \;\; b(a-1) \geqslant 0$

$= a^2 + \left(b - (a-1)K\right)^2, \, b\,(a-1) \leqslant 0$ (why? try to figure out it.).

And, $\inf\limits_{\{a,b|b(a-1)\geqslant 0\}} \sup\limits_{-K\leqslant\theta\leqslant K} E_\theta \left(T\left(x\right) - \theta\right)^2 = \inf\limits_{\{a,b|b(a-1)\geqslant 0\}} a^2 + \left(b + (a-1)\,K\right)^2$

$= \inf\limits_{a\in R} a^2 + \left(a-1\right)^2 K^2$ (why does b need to be 0 ?)

$= (\dfrac{K^2}{K^2+1})^2 + \left(\dfrac{K^2}{K^2+1} - 1\right)^2 K^2$

(why? $\dfrac{d}{da}\left(a^2 + \left(a-1\right)^2 K^2\right) = 0.$)

$= \dfrac{K^2}{K^2+1},$

Similarly, we have $\inf\limits_{\{a,b|b(a-1)\geqslant 0\}} \sup\limits_{-K\leqslant\theta\leqslant K} E_\theta \left(T\left(x\right) - \theta\right)^2 = \dfrac{K^2}{K^2+1}.$

Thus our minimax linear estimator is $T_{LM}\left(X\right) = \dfrac{K^2}{K^2+1} X.$

(ii) from (i), we know that $E_\theta \left(T_{LM}(x) - \theta\right)^2 = \dfrac{K^2}{K^2+1} < 1 = E_\theta \left(X - \theta\right)^2$.

(iii) The Beyesian estimator is $\dfrac{\tau^2}{\tau^2+1}X$ in the unrestricted model $\theta = R$ wirh the prior distribution $N(0, \tau^2)$. Thus $\tau^2 = K^2$.

Exercise H4.2 Let $X_1, \ldots, X_n$ be independent and identically distributed with Poisson law Po(λ), where $\lambda > 0$ is unknown. Assume that the statement of Theorem 4.2 (Cramer-Rao bound for i.i.d. data) is valid in this case (the sample space $\mathcal{X}$ is not finite here but countable, but it will be shown in lectures that the Cramer-Rao bound (4.10) is also valid here) .

(i) Compute the Fisher information $I_F(\lambda)$ for one observation X_1.

(ii) Show that for n observations, the sample mean $\bar{X}_n$ is a uniformly best unbiased estimator.

Solution: (i) since $P_\lambda(x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$, we have

$$\begin{aligned}
\frac{d}{d\lambda} l_\lambda(x) &= \frac{d}{d\lambda} \ln P_\lambda(x) \\
&= \frac{d}{d\lambda}\left(-\lambda + x\ln\lambda - \ln x!\right) \\
&= -1 + \frac{x}{\lambda},
\end{aligned}$$

then,

$$\begin{aligned}
I_F(\lambda) &= E_\lambda \left(\frac{d}{d\lambda} l_\lambda(X)\right)^2 \\
&= E_\lambda \left(-1 + \frac{X}{\lambda}\right)^2 \\
&= \frac{1}{\lambda^2} E_\lambda (X - \lambda)^2 \\
&= \frac{1}{\lambda^2} var(X) \\
&= \frac{1}{\lambda} .
\end{aligned}$$

(ii) Note that $E_\lambda \left(\bar{X_n}\right) = \dfrac{1}{n} n\lambda = \lambda,$

and,

$$Var\left(\bar{X_n}\right) = \frac{1}{n} Var(X) = \frac{\lambda}{n} = \frac{1}{n I_F(\lambda)}.$$

This implies $\bar{X_n}$ is a uniformly best unbiased estimator.

Exercise H4.3 Consider the **Cramer-Rao bound in a continuous statistical model**. Assume $X_1, \ldots, X_n$ are i.i.d. random variables, each having a density $p_\vartheta(x)$ defined on $\mathbb{R}$, where $\vartheta \in \Theta$ is unknown and Θ is an interval in $\mathbb{R}$. Consider the following (analogous) definition of a Fisher information $I_F(\vartheta)$ for X_1:

$$I_F(\vartheta) = E_\vartheta \left(\frac{\partial}{\partial \vartheta} \log p_\vartheta(X_1) \right)^2$$

(where it is assumed that all expressions are well defined, i.e. $\log p_\vartheta(x)$ is defined for all x, is differentiable in ϑ, and the expectation above is finite). Assume again that the Cramer-Rao bound (4.10) is also valid here, for all unbiased estimators of ϑ.

(i) Specializing to Model II (Gaussian location model; $p_\vartheta(x)$ is the density of $N(\vartheta, \sigma^2)$ with unknown $\vartheta \in \mathbb{R}$ and known $\sigma^2 > 0$), compute $I_F(\vartheta)$ for one observation X_1.

(ii) Show that for n observations in Model II, the sample mean $\bar{X}_n$ is a uniformly best unbiased estimator.

(iii) Specialize to the **Gaussian scale model:** $p_\vartheta(x)$ is the density of $N(0, \sigma^2)$, with unknown $\vartheta = \sigma^2 > 0$; compute $I_F(\sigma^2)$ for one observation X_1.

(iv) Show that for n observations in the Gaussian scale model, the sample variance

$$S^2 = n^{-1} \sum_{i=1}^{n} X_i^2$$

is a uniformly best unbiased estimator. *(See next page for hints)*

Hints: **(a)** Note that σ^2 is treated as parameter, not σ; so it may be convenient to write ϑ for σ^2 when taking derivatives.

(b) Note that

$$\mathrm{Var}_{\sigma^2} X_1^2 = 2\sigma^4.$$

A short proof runs as follows. We have $X_1 = \sigma Y$ for standard normal Z, so it suffices to prove $\mathrm{Var} Z^2 = 2$. Now $\mathrm{Var} Z^2 = EZ^4 - (EZ^2)^2$, so it suffices to prove $EZ^2 = 3$. For the standard normal density φ we have by partial integration, using $\varphi'(x) = -x\varphi(x)$

$$\int x^4 \varphi(x) dx = -\int x^3 \varphi'(x) dx = 3 \int x^2 \varphi(x) dx = 3.$$

Solution: (i) since $P_\theta(x) = \dfrac{1}{(2\pi)^{\frac{1}{2}} \sigma} e^{-\dfrac{(x-\theta)^2}{2\sigma^2}}$, we have

$$\begin{aligned} \frac{d}{d\theta} l_\theta(x) &= \frac{d}{d\theta} \left(-\ln\left((2\pi)^{\frac{1}{2}} \sigma \right) - \frac{(x-\theta)^2}{2\sigma^2} \right) \\ &= \frac{x-\theta}{\sigma^2}, \end{aligned}$$

then,

$$
\begin{aligned}
I_F(\theta) &= E_\theta\left(\frac{d}{d\theta} l_\theta(X)\right)^2 \\
&= E_\theta\left(\frac{X-\theta}{\sigma^2}\right)^2 \\
&= \frac{1}{\sigma^4} var(X) \\
&= \frac{1}{\sigma^2}.
\end{aligned}
$$

(ii) Note that $E_\theta\left(\bar{X_n}\right) = \theta$, and $Var\left(\bar{X_n}\right) = \frac{1}{n}\ var(X) = \frac{\sigma^2}{n} = \frac{1}{nI_F(\theta)}$.

This implie $\bar{X_n}$ is a uniformly best unbiased estimator.

(iii) since $\quad P_{\sigma^2}(x) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma} e^{-\frac{x^2}{2\sigma^2}}$, we have

$$
\begin{aligned}
\frac{d}{d\sigma^2}\ l_{\sigma^2}(x) &= \frac{d}{d\sigma^2}\left(-\ln(2\pi)^{\frac{1}{2}} - \frac{1}{2}\ln\sigma^2 - \frac{x^2}{2\sigma^2}\right) \\
&= -\frac{1}{2\sigma^2} + \frac{x^2}{2\sigma^4},
\end{aligned}
$$

then,

$$
\begin{aligned}
I_F\left(\sigma^2\right) &= E_{\sigma^2}\left(\frac{d}{d\sigma^2}\ l_{\sigma^2}(X)\right)^2 \\
&= E_{\sigma^2}\left(-\frac{1}{2\sigma^2} + \frac{X^2}{2\sigma^4}\right)^2 \\
&= \left(-\frac{1}{2\sigma^2}\right)^2 + 2\left(-\frac{1}{2\sigma^2}\right)\frac{E_{\sigma^2}(X^2)}{2\sigma^4} + \frac{E_{\sigma^2}(X^4)}{(2\sigma^4)^2} \\
&= \left(-\frac{1}{2\sigma^2}\right)^2 + 2\left(-\frac{1}{2\sigma^2}\right)\frac{\sigma^2}{2\sigma^4} + \frac{2\sigma^4 + (\sigma^2)^2}{(2\sigma^4)^2} \\
&= \frac{1}{2\sigma^4}.
\end{aligned}
$$

(from the hint(b), we have $E\left(X^4\right) = Var\left(X^2\right)$ $+\left(E\left(X^2\right)\right)^2$.)

(iv) Note that $E\left(S^2\right)\ = \frac{1}{n} nE\left(X^2\right)\ = E\left(X^2\right) = \sigma^2$, and

$$
var\left(S^2\right)\ = \frac{1}{n} var\left(X^2\right)
$$

$$
\begin{aligned}
&= \frac{1}{n} E \left(X^2 - EX^2 \right)^2 \\
&= \frac{1}{n} \left(EX^4 - 2\sigma^2 EX^2 + \sigma^4 \right) \\
&= \frac{1}{n} (3\sigma^4 - 2\sigma^2\sigma^2 + \sigma^4) \\
&= \frac{2\sigma^4}{n} \\
&= \frac{1}{nI_F\left(\sigma^2\right)}.
\end{aligned}
$$

This implie S^2 is a uniformly best unbiased estimator.