Measure Theory

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1 Introduction

We always denote by X our *universe*, i.e. all the sets we shall consider are subsets of X.

Recall some standard notation. 2^X everywhere denotes the set of all subsets of a given set X. If $A \cap B = \varnothing$ then we often write $A \sqcup B$ rather than $A \cup B$, to underline the disjointness. The complement (in X) of a set A is denoted by A^c. By $A \triangle B$ the *symmetric difference* of A and B is denoted, i.e. $A \triangle B = (A \setminus B) \cup (B \setminus A)$. Letters $i,\ j,\ k$ always denote positive integers. The sign $\restriction$ is used for restriction of a function (operator etc.) to a subset (subspace).

1.1 The Riemann integral

Recall how to construct the Riemannian integral. Let $f : [a, b] \to \mathbb{R}$. Consider a partition π of $[a, b]$:

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

and set $\Delta x_k = x_{k+1} - x_k$, $|\pi| = \max\{\Delta x_k : k = 0, 1, \ldots, n-1\}$, $m_k = \inf\{f(x) : x \in [x_k, x_{k+1}]\}$, $M_k = \sup\{f(x) : x \in [x_k, x_{k+1}]\}$. Define the upper and lower Riemann—Darboux sums

$$\underline{s}(f, \pi) = \sum_{k=0}^{n-1} m_k \Delta x_k, \quad \bar{s}(f, \pi) = \sum_{k=0}^{n-1} M_k \Delta x_k.$$

One can show (the Darboux theorem) that the following limits exist

$$\lim_{|\pi| \to 0} \underline{s}(f, \pi) = \sup_{\pi} \underline{s}(f, \pi) = \underline{\int_a^b} f dx$$

$$\lim_{|\pi| \to 0} \bar{s}(f, \pi) = \inf_{\pi} \bar{s}(f, \pi) = \overline{\int_a^b} f dx.$$

Clearly,

$$\underline{s}(f,\pi) \le \underline{\int_a^b} f dx \le \overline{\int_a^b} f dx \le \bar{s}(f,\pi)$$

for any partition π.

The function f is said to be Riemann integrable on $[a,b]$ if the upper and lower integrals are equal. The common value is called Riemann integral of f on $[a,b]$.

The functions cannot have a large set of points of discontinuity. More presicely this will be stated further.

1.2 The Lebesgue integral

It allows to integrate functions from a much more general class. First, consider a very useful example. For $f, g \in C[a,b]$, two continuous functions on the segment $[a,b] = \{x \in \mathbb{R} : a \leqslant x \leqslant b\}$ put

$$\rho_1(f,g) = \max_{a \leqslant x \leqslant b} |f(x) - g(x)|,$$

$$\rho_2(f,g) = \int_a^b |f(x) - g(x)| \mathrm{d}x.$$

Then $(C[a,b], \rho_1)$ is a complete metric space, when $(C[a,b], \rho_2)$ is not. To prove the latter statement, consider a family of functions $\{\varphi_n\}_{n=1}^{\infty}$ as drawn on Fig.1. This is a Cauchy sequence with respect to ρ_2. However, the limit does not belong to $C[a,b]$.

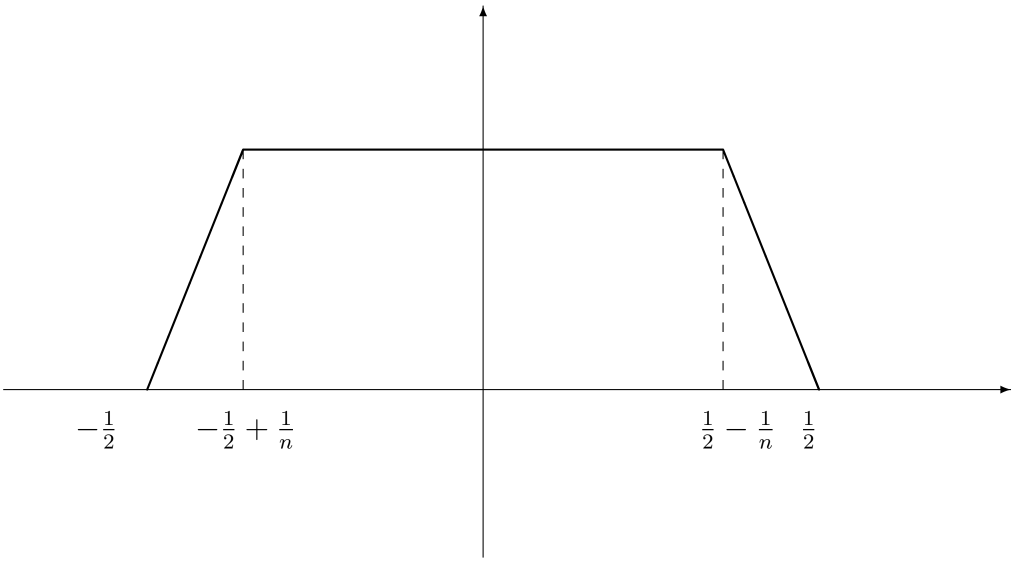


Figure 1: The function φ_n.

2 Systems of Sets

Definition 2.1 A ring of sets *is a non-empty subset in* 2^X *which is closed with respect to the operations* $\cup$ *and* $\setminus$.

Proposition. Let $\mathfrak{K}$ be a ring of sets. Then $\varnothing \in \mathfrak{K}$.

Proof. Since $\mathfrak{K} \neq \varnothing$, there exists $A \in \mathfrak{K}$. Since $\mathfrak{K}$ contains the difference of every two its elements, one has $A \setminus A = \varnothing \in \mathfrak{K}$. ■

Examples.

1. The two extreme cases are $\mathfrak{K} = \{\varnothing\}$ and $\mathfrak{K} = 2^X$.
2. Let $X = \mathbb{R}$ and denote by $\mathfrak{K}$ all finite unions of semi-segments $[a, b)$.

Definition 2.2 A semi-ring *is a collection of sets* $\mathfrak{P} \subset 2^X$ *with the following properties:*

1. *If* $A, B \in \mathfrak{P}$ *then* $A \cap B \in \mathfrak{P}$;

2. *For every $A, B \in \mathfrak{P}$ there exists a finite disjoint collection (C_j) $j = 1, 2, \ldots, n$ of sets (i.e. $C_i \cap C_j = \varnothing$ if $i \neq j$) such that*

$$A \setminus B = \bigsqcup_{j=1}^{n} C_j.$$

Example. Let $X = \mathbb{R}$, then the set of all semi-segments, $[a, b)$, forms a semi-ring.

Definition 2.3 An algebra (of sets) *is a ring of sets containing $X \in 2^X$.*

Examples.

1. $\{\varnothing, X\}$ and 2^X are the two extreme cases (note that they are different from the corresponding cases for rings of sets).
2. Let $X = [a, b)$ be a fixed interval on $\mathbb{R}$. Then the system of finite unions of subintervals $[\alpha, \beta) \subset [a, b)$ forms an algebra.
3. The system of all bounded subsets of the real axis is a ring (*not an algebra*).

Remark. $\mathfrak{A}$ is algebra if (i) $A, B \in \mathfrak{A} \Longrightarrow A \cup B \in \mathfrak{A}$, (ii) $A \in \mathfrak{A} \Longrightarrow A^c \in \mathfrak{A}$.

Indeed, 1) $A \cap B = (A^c \cup B^c)^c$; 2) $A \setminus B = A \cap B^c$.

Definition 2.4 A σ-ring (a σ-algebra) *is a ring (an algebra) of sets which is closed with respect to all countable unions.*

Definition 2.5 A ring (an algebra, a σ-algebra) of sets, $\mathfrak{K}(\mathfrak{U})$ *generated by a collection of sets $\mathfrak{U} \subset 2^X$ is the minimal ring (algebra, σ-algebra) of sets containing $\mathfrak{U}$.*

In other words, it is the intersection of all rings (algebras, σ-algebras) of sets containing $\mathfrak{U}$.

3 Measures

Let X be a set, $\mathfrak{A}$ an algebra on X.

Definition 3.1 *A function μ: $\mathfrak{A} \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ is called a* measure *if*

1. *$\mu(A) \geqslant 0$ for any $A \in \mathfrak{A}$ and $\mu(\varnothing) = 0$;*
2. *if $(A_i)_{i\geqslant 1}$ is a disjoint family of sets in $\mathfrak{A}$ ($A_i \cap A_j = \varnothing$ for any $i \neq j$) such that $\bigsqcup_{i=1}^{\infty} A_i \in \mathfrak{A}$, then*
$$\mu(\bigsqcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

The latter important property, is called *countable additivity* or *σ-additivity* of the measure μ.

Let us state now some elementary properties of a measure. Below till the end of this section $\mathfrak{A}$ is an algebra of sets and μ is a measure on it.

1. (Monotonicity of μ) If $A, B \in \mathfrak{A}$ and $B \subset A$ then $\mu(B) \leqslant \mu(A)$.

   Proof. $A = (A \setminus B) \sqcup B$ implies that
$$\mu(A) = \mu(A \setminus B) + \mu(B).$$
   Since $\mu(A \setminus B) \geq 0$ it follows that $\mu(A) \geq \mu(B)$.

2. (Subtractivity of μ). If $A, B \in \mathfrak{A}$ and $B \subset A$ and $\mu(B) < \infty$ then $\mu(A \setminus B) = \mu(A) - \mu(B)$.

   Proof. In 1) we proved that
$$\mu(A) = \mu(A \setminus B) + \mu(B).$$
   If $\mu(B) < \infty$ then
$$\mu(A) - \mu(B) = \mu(A \setminus B).$$

3. If $A, B \in \mathfrak{A}$ and $\mu(A \cap B) < \infty$ then $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$.

   Proof. $A \cap B \subset A,\ A \cap B \subset B$, therefore
$$A \cup B = (A \setminus (A \cap B)) \sqcup B.$$
   Since $\mu(A \cap B) < \infty$, one has
$$\mu(A \cup B) = (\mu(A) - \mu(A \cap B)) + \mu(B).$$

4. (Semi-additivity of μ). If $(A_i)_{i\geq 1} \subset \mathfrak{A}$ such that $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{A}$ then

$$\mu(\bigcup_{i=1}^{\infty} A_i) \leqslant \sum_{i=1}^{\infty} \mu(A_i).$$

Proof. First let us proove that

$$\mu(\bigcup_{i=1}^{n} A_i) \leqslant \sum_{i=1}^{n} \mu(A_i).$$

Note that the family of sets

$$\begin{aligned} B_1 &= A_1 \\ B_2 &= A_2 \setminus A_1 \\ B_3 &= A_3 \setminus (A_1 \cup A_2) \\ &\dots \\ B_n &= A_n \setminus \bigcup_{i=1}^{n-1} A_i \end{aligned}$$

is disjoint and $\bigsqcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i$. Moreover, since $B_i \subset A_i$, we see that $\mu(B_i) \leq \mu(A_i)$. Then

$$\mu(\bigcup_{i=1}^{n} A_i) = \mu(\bigsqcup_{i=1}^{n} B_i) = \sum_{i=1}^{n} \mu(B_i) \leq \sum_{i=1}^{n} \mu(A_i).$$

Now we can repeat the argument for the infinite family using σ-additivity of the measure.

3.1 Continuity of a measure

Theorem 3.1 *Let $\mathfrak{A}$ be an algebra, $(A_i)_{i\geq 1} \subset \mathfrak{A}$ a monotonically increasing sequence of sets ($A_i \subset A_{i+1}$) such that $\bigcup_{i\geq 1} \in \mathfrak{A}$. Then*

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mu(A_n).$$

Proof. 1). If for some n_0 $\mu(A_{n_0}) = +\infty$ then $\mu(A_n) = +\infty$ $\forall n \geq n_0$ and $\mu(\bigcup_{i=1}^{\infty} A_i) = +\infty$.

2). Let now $\mu(A_i) < \infty$ $\forall i \geq 1$.

Then

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \mu(A_1 \sqcup (A_2 \setminus A_1) \sqcup \ldots \sqcup (A_n \setminus A_{n-1}) \sqcup \ldots)$$

$$= \mu(A_1) + \sum_{k=2}^{\infty} \mu(A_k \setminus A_{k-1})$$

$$= \mu(A_1) + \lim_{n\to\infty} \sum_{k=2}^{n} (\mu(A_k) - \mu(A_{k-1})) = \lim_{n\to\infty} \mu(A_n).$$

3.2 Outer measure

Let a be an algebra of subsets of X and μ a measure on it. Our purpose now is to extend μ to as many elements of 2^X as possible.

An arbitrary set $A \subset X$ can be always covered by sets from $\mathfrak{A}$, i.e. one can always find $E_1, E_2, \ldots \in \mathfrak{A}$ such that $\bigcup_{i=1}^{\infty} E_i \supset A$. For instance, $E_1 = X, E_2 = E_3 = \ldots = \varnothing$.

Definition 3.2 *For $A \subset X$ its* outer measure *is defined by*

$$\mu^*(A) = \inf \sum_{i=1}^{\infty} \mu(E_i)$$

where the infimum is taken over all $\mathfrak{A}$-coverings of the set A, i.e. all collections (E_i), $E_i \in \mathfrak{A}$ with $\bigcup_i E_i \supset A$.

Remark. The outer measure always exists since $\mu(A) \geqslant 0$ for every $A \in \mathfrak{A}$.

Example. Let $X = \mathbb{R}^2$, $\mathfrak{A} = \mathfrak{K}(\mathfrak{P})$, -$\sigma$-algebra generated by $\mathfrak{P}$, $\mathfrak{P} = \{[a, b) \times \mathbb{R}^1\}$. Thus $\mathfrak{A}$ consists of countable unions of strips like one drawn on the picture. Put $\mu([a, b) \times \mathbb{R}^1) = b - a$. Then, clearly, the outer measure of the unit disc $x^2 + y^2 \leqslant 1$ is equal to 2. The same value is for the square $|x| \leqslant 1$, $|y| \leqslant 1$.

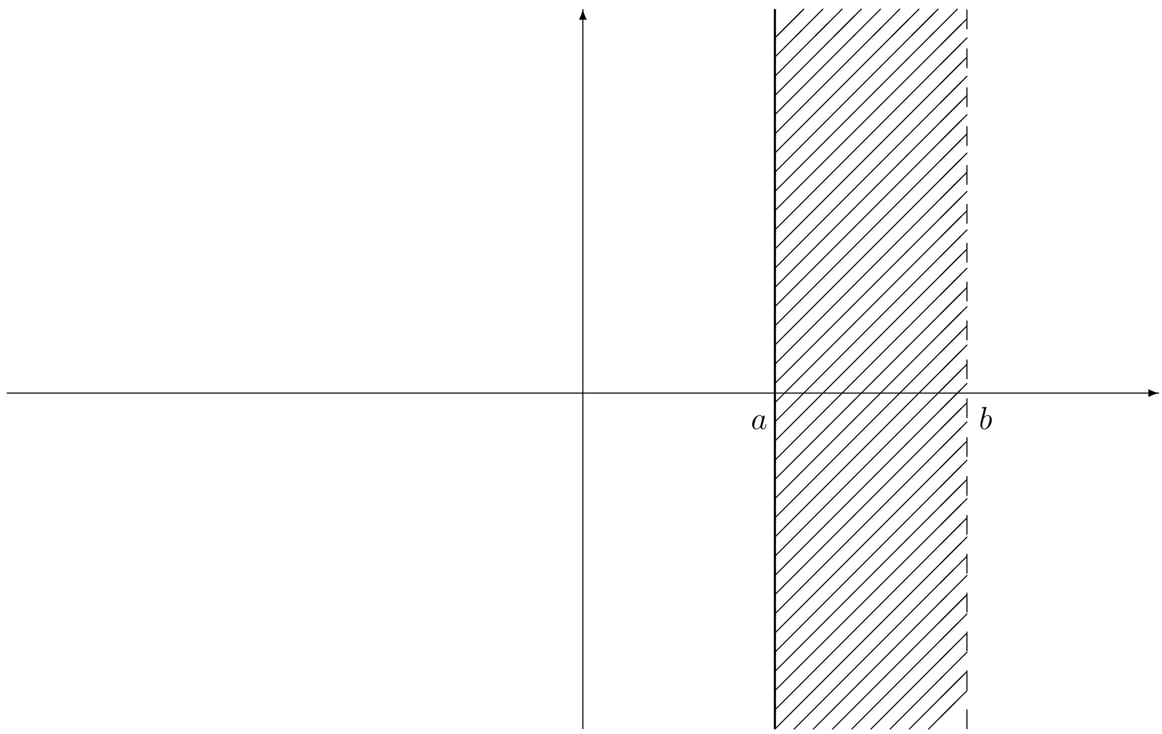
Theorem 3.2 *For $A \in \mathfrak{A}$ one has $\mu^*(A) = \mu(A)$.*

In other words, μ^* is an extension of μ.

Proof. 1. A is its own covering. This implies $\mu^*(A) \leqslant \mu(A)$.

2. By definition of infimum, for any $\varepsilon > 0$ there exists a $\mathfrak{A}$-covering (E_i) of A such that $\sum_i \mu(E_i) < \mu^*(A) + \varepsilon$. Note that

$$A = A \cap (\bigcup_i E_i) = \bigcup_i (A \cap E_i).$$



Using consequently σ-semiadditivity and monotonicity of μ, one obtains:

$$\mu(A) \leqslant \sum_i \mu(A \cap E_i) \leqslant \sum_i \mu(E_i) < \mu^*(A) + \varepsilon.$$

Since ε is arbitrary, we conclude that $\mu(A) \leqslant \mu^*(A)$. ■

It is evident that $\mu^*(A) \geqslant 0, \quad \mu^*(\varnothing) = 0$ (Check !).

Lemma. Let $\mathfrak{A}$ be an algebra of sets (not necessary σ-algebra), μ a measure on $\mathfrak{A}$. If there exists a set $A \in \mathfrak{A}$ such that $\mu(A) < \infty$, then $\mu(\varnothing) = 0$.

Proof. $\mu(A \setminus A) = \mu(A) - \mu(A) = 0$. ■

Therefore the property $\mu(\varnothing) = 0$ can be substituted with the existence in $\mathfrak{A}$ of a set with a finite measure.

Theorem 3.3 *(Monotonicity of outer measure). If $A \subset B$ then $\mu^*(A) \leqslant \mu^*(B)$.*

Proof. Any covering of B is a covering of A. ■

Theorem 3.4 *(σ-semiadditivity of μ^*). $\mu^*(\bigcup_{j=1}^{\infty} A_j) \leqslant \sum_{j=1}^{\infty} \mu^*(A_j)$.*

Proof. If the series in the right-hand side diverges, there is nothing to prove. So assume that it is convergent.

By the definition of outer measur for any $\varepsilon > 0$ and for any j there exists an $\mathfrak{A}$-covering $\bigcup_k E_{kj} \supset A_j$ such that

$$\sum_{k=1}^{\infty} \mu(E_{kj}) < \mu^*(A_j) + \frac{\varepsilon}{2^j}.$$

Since

$$\bigcup_{j,k=1}^{\infty} E_{kj} \supset \bigcup_{j=1}^{\infty} A_j,$$

the definition of μ^* implies

$$\mu^*(\bigcup_{j=1}^{\infty} A_j) \leqslant \sum_{j,k=1}^{\infty} \mu(E_{kj})$$

and therefore

$$\mu^*(\bigcup_{j=1}^{\infty} A_j) < \sum_{j=1}^{\infty} \mu^*(A_j) + \varepsilon.$$

■

3.3 Measurable Sets

Let $\mathfrak{A}$ be an algebra of subsets of X, μ a measure on it, μ^* the outer measure defined in the previous section.

Definition 3.3 *$A \subset X$ is called* a measurable set (by Carathèodory) *if for any $E \subset X$ the following relation holds:*

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Denote by $\tilde{\mathfrak{A}}$ the collection of all set which are measurable by Carathèodory and set $\tilde{\mu} = \mu^* \restriction \tilde{\mathfrak{A}}$.

Remark Since $E = (E \cap A) \cup (E \cap A^c)$, due to semiadditivity of the outer measure

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Theorem 3.5 *$\tilde{\mathfrak{A}}$ is a σ-algebra containing $\mathfrak{A}$, and $\tilde{\mu}$ is a measure on $\tilde{\mathfrak{A}}$.*

Proof. We devide the proof into several steps.

1. <u>If</u> $A, B \in \tilde{\mathfrak{A}}$ <u>then</u> $A \cup B \in \tilde{\mathfrak{A}}$.

By the definition one has

$$\mu^*(E) = \mu^*(E \cap B) + \mu^*(E \cap B^c). \tag{1}$$

Take $E \cap A$ instead of E:

$$\mu^*(E \cap A) = \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c). \tag{2}$$

Then put $E \cap A^c$ in (1) instead of E

$$\mu^*(E \cap A^c) = \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c). \tag{3}$$

Add (2) and (3):

$$\mu^*(E) = \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c). \tag{4}$$

Substitute $E \cap (A \cup B)$ in (4) instead of E. Note that

$$\begin{array}{ll} 1) & E \cap (A \cup B) \cap A \cap B = E \cap A \cap B \\ 2) & E \cap (A \cup B) \cap A^c \cap B = E \cap A^c \cap B \\ 3) & E \cap (A \cup B) \cap A \cap B^c = E \cap A \cap B^c \\ 4) & E \cap (A \cup B) \cap A^c \cap B^c = \varnothing. \end{array}$$

One has

$$\mu^*(E \cap (A \cup B)) = \mu^*(E \cap A \cap B) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A \cap B^c). \tag{5}$$

From (4) and (5) we have

$$\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c).$$

2. <u>If</u> $A \in \tilde{\mathfrak{A}}$ <u>then</u> $A^c \in \tilde{\mathfrak{A}}$.

The definition of measurable set is symmetric with respect to A and A^c.

Therefore $\tilde{\mathfrak{A}}$ is an algebra of sets.

3.

Let $A, B \in \mathfrak{A}$, $A \cap B = \varnothing$. From (5)

$$\mu^*(E \cap (A \sqcup B)) = \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A \cap B^c) = \mu^*(E \cap B) + \mu^*(E \cap A).$$

4. $\tilde{\mathfrak{A}}$ is a σ-algebra.

From the previous step, by induction, for any finite disjoint collection (B_j) of sets:

$$\mu^*(E \cap (\bigsqcup_{j=1}^{n} B_j)) = \sum_{j=1}^{n} \mu^*(E \cap B_j). \tag{6}$$

Let $A = \bigcup_{j=1}^{\infty} A_j, A_j \in \mathfrak{A}$. Then $A = \bigcup_{j=1}^{\infty} B_j,\ B_j = A_j \setminus \bigcup_{k=1}^{j-1} A_k$ and $B_i \cap B_j = \varnothing$ $(i \neq j)$. It suffices to prove that

$$\mu^*(E) \geqslant \mu^*(E \cap (\bigsqcup_{j=1}^{\infty} B_j)) + \mu^*(E \cap (\bigsqcup_{j=1}^{\infty} B_j)^c). \tag{7}$$

Indeed, we have already proved that μ^* is σ-semi-additive.

Since $\tilde{\mathfrak{A}}$ is an algebra, it follows that $\bigsqcup_{j=1}^{n} B_j \in \tilde{\mathfrak{A}} (\forall n \in \mathbb{N})$ and the following inequality holds for every n:

$$\mu^*(E) \geqslant \mu^*(E \cap (\bigsqcup_{j=1}^{n} B_j)) + \mu^*(E \cap (\bigsqcup_{j=1}^{n} B_j)^c). \tag{8}$$

Since $E \cap (\bigsqcup_{j=1}^{\infty} B_j)^c \subset E \cap (\bigsqcup_{j=1}^{n} B_j)^c$, by monotonicity of the mesasure and (8)

$$\mu^*(E) \geq \sum_{j=1}^{n} \mu^*(E \cap B_j) + \mu^*(E \cap A^c). \tag{9}$$

Passing to the limit we get

$$\mu^*(E) \geq \sum_{j=1}^{\infty} \mu^*(E \cap B_j) + \mu^*(E \cap A^c). \tag{10}$$

Due to semiadditivity

$$\mu^*(E \cap A) = \mu^*(E \cap (\bigsqcup_{j=1}^{\infty} B_j)) = \mu^*(\bigsqcup_{j=1}^{\infty} (E \cap B_j)) \leq \sum_{j=1}^{\infty} \mu^*(E \cap B_j).$$

Compare this with (10):

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Thus, $A \in \tilde{\mathfrak{A}}$, which means that $\tilde{\mathfrak{A}}$ is a σ-algebra.

5. $\tilde{\mu} = \mu^* \restriction \tilde{\mathfrak{A}}$ is a measure.

We need to prove only σ-additivity. Let $E = \bigsqcup_{j=1}^{\infty} A_j$. From(10) we get

$$\mu^*(\bigsqcup_{j=1}^{\infty} A_j) \geqslant \sum_{j=1}^{\infty} \mu^*(A_j).$$

The oposite inequality follows from σ-semiadditivity of μ^*.

6. $\tilde{\mathfrak{A}} \supset \mathfrak{A}$.

Let $A \in \mathfrak{A}$, $E \subset X$. We need to prove:

$$\mu^*(E) \geqslant \mu^*(E \cap A) + \mu^*(E \cap A^c). \tag{11}$$

If $E \in \mathfrak{A}$ then (11) is clear since $E \cap A$ and $E \cap A^c$ are disjoint and both belong to $\mathfrak{A}$ where $\mu^* = \mu$ and so is additive.

For $E \subset X$ for $\forall \varepsilon > 0$ there exists a $\mathfrak{A}$-covering (E_j) of E such that

$$\mu^*(E) + \varepsilon > \sum_{j=1}^{\infty} \mu(E_j). \tag{12}$$

Now, since $E_j = (E_j \cap A) \cup (E_j \cap A^c)$, one has

$$\mu(E_j) = \mu(E_j \cap A) + \mu(E_j \cap A)$$

and also

$$E \cap A \subset \bigcup_{j=1}^{\infty} (E_j \cap A)$$

$$E \cap A^c \subset \bigcup_{j=1}^{\infty} (E_j \cap A^c)$$

By monotonicity and σ-semiadditivity

$$\mu^*(E \cap A) \leqslant \sum_{j=1}^{\infty} \mu(E_j \cap A),$$

$$\mu^*(E \cap A^c) \leqslant \sum_{j=1}^{\infty} \mu(E_j \cap A^c).$$

Adding the last two inequalities we obtain

$$\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \sum_{j=1}^{\infty} \mu^*(E_j) < \mu^*(E) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, (11) is proved. ∎

The following theorem is a direct consequence of the previous one.

Theorem 3.6 *Let $\mathfrak{A}$ be an algebra of subsets of X and μ be a measure on it. Then there exists a σ-algebra $\mathfrak{A}_1 \supset \mathfrak{A}$ and a measure μ_1 on $\mathfrak{A}_1$ such that $\mu_1 \restriction \mathfrak{A} = \mu$.*

Remark. Consider again an algebra $\mathfrak{A}$ of subsets of X. Denot by $\mathfrak{A}_\sigma$ the generated σ-algebra and construct the extension μ_σ of μ on $\mathfrak{A}_\sigma$. This extension is called *minimal extension of measure.*

Since $\tilde{\mathfrak{A}} \supset \mathfrak{A}$ therefore $\mathfrak{A}_\sigma \subset \tilde{\mathfrak{A}}$. Hence one can set $\mu_\sigma = \tilde{\mu} \restriction \mathfrak{A}_\sigma$. Obviously μ_σ is a minimal extension of μ. It always exists. On can also show (see below) that this extension is unique.

Theorem 3.7 *Let μ be a measure on an algebra $\mathfrak{A}$ of subsets of X, μ^* the corresponding outer measure. If $\mu^*(A) = 0$ for a set $A \subset X$ then $A \in \tilde{\mathfrak{A}}$ and $\tilde{\mu}(A) = 0$.*

Proof. Clearly, it suffices to prove that $A \in \tilde{\mathfrak{A}}$. Further, it suffices to prove that $\mu^*(E) \geqslant \mu^*(E \cap A) + \mu^*(E \cap A^c)$. The latter statement follows from monotonicity of μ^*. Indeed, one has $\mu^*(E \cap A) \leqslant \mu^*(A) = 0$ and $\mu^*(E \cap A^c) \leqslant \mu^*(E)$. ■

Definition 3.4 *A measure μ on an algebra of sets $\mathfrak{A}$ is called* complete *if conditions $B \subset A, \quad A \in \mathfrak{A}, \quad \mu(A) = 0$ imply $B \in \mathfrak{A}$ and $\mu(B) = 0$.*

Corollary. $\tilde{\mu}$ is a complete measure.

Definition 3.5 *A measure μ on an algebra $\mathfrak{A}$ is called finite if $\mu(X) < \infty$. It is called σ-finite if the is an increasing sequence $(F_j)_{j\geq 1} \subset \mathfrak{A}$ such that $X = \bigcup_j F_j$ and $\mu(F_j) < \infty$ $\forall j$.*

Theorem 3.8 *Let μ be a σ-finite measure on an algebra $\mathfrak{A}$. Then there exist a unique extension of μ to a measure on $\tilde{\mathfrak{A}}$.*

Proof. It suffices to sjow uniqueness. Let ν be another extension of μ ($\nu \restriction \mathfrak{A} = \mu \restriction \mathfrak{A}$).

First, let μ (and therefore ν, μ^*) be finite. Let $A \in \tilde{\mathfrak{A}}$. Let $(E_j) \subset \mathfrak{A}$ such that $A \subset \bigcup_j E_j$. We have

$$\nu(A) \leq \nu(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} \nu(E_j) = \sum_{j=1}^{\infty} \mu(E_j).$$

Therefore

$$\nu(A) \leq \mu^*(A) \quad \forall A \in \tilde{\mathfrak{A}}.$$

Since μ^* and ν are additive (on $\tilde{\mathfrak{A}}$) it follows that

$$\mu^*(A) + \mu^*(A^c) = \nu(A) + \nu(A^c).$$

The terms in the RHS are finite and $\nu(A) \le \mu^*(A),\ \nu(A^c) \le \mu^*(A^c)$. From this we infer that

$$\nu(A) = \mu^*(A) \ \ \forall A \in \tilde{\mathfrak{A}}.$$

Now let μ be σ-finite, (F_j) be an increasing sequence of sets from $\mathfrak{A}$ such that $\mu(F_j) < \infty\ \forall j$ and $X = \bigcup_{j=1}^{\infty} F_j$. From what we have already proved it follows that

$$\mu^*(A \cap F_j) = \nu(A \cap F_j)\ \forall A \in \tilde{\mathfrak{A}}.$$

Therefore

$$\mu^*(A) = \lim_j \mu^*(A \cap F_j) = \lim_j \nu(A \cap F_j) = \nu(A). \quad \blacksquare$$

Theorem 3.9 *(Continuity of measure). Let $\mathfrak{A}$ be a σ-algebra with a measure μ, $\{A_j\} \subset \mathfrak{A}$ a monotonically increasing sequence of sets. Then*

$$\mu(\bigcup_{j=1}^{\infty} A_j) = \lim_{j\to\infty} \mu(A_j).$$

Proof. One has:

$$A = \bigcup_{j=1}^{\infty} A_j = \bigsqcup_{j=2}^{\infty} (A_{j+1} \setminus A_j) \sqcup A_1.$$

Using σ-additivity and subtractivity of μ,

$$\mu(A) = \sum_{j=1}^{\infty} (\mu(A_{j+1}) - \mu(A_j)) + \mu(A_1) = \lim_{j\to\infty} \mu(A_j). \quad \blacksquare$$

Similar assertions for a decreasing sequence of sets in $\mathfrak{A}$ can be proved using de Morgan formulas.

Theorem 3.10 *Let $A \in \tilde{\mathfrak{A}}$. Then for any $\varepsilon > 0$ there exists $A_\varepsilon \in \mathfrak{A}$ such that $\mu^*(A \triangle A_\varepsilon) < \varepsilon$.*

Proof. 1. For any $\varepsilon > 0$ there exists an $\mathfrak{A}$ cover $\bigcup E_j \supset A$ such that

$$\sum_j \mu(E_j) < \mu^*(A) + \frac{\varepsilon}{2} = \tilde{\mu}(A) + \frac{\varepsilon}{2}.$$

On the other hand,

$$\sum_j \mu(E_j) \geqslant \tilde{\mu}(\bigcup_j E_J).$$

The monotonicity of $\tilde{\mu}$ implies

$$\tilde{\mu}(\bigcup_{j=1}^{\infty} E_J) = \lim_{n\to\infty} \tilde{\mu}(\bigcup_{j=1}^{n} E_j),$$

hence there exists a positive integer N such that

$$\tilde{\mu}(\bigcup_{j=1}^{\infty} E_j) - \tilde{\mu}(\bigcup_{j=1}^{N} E_j) < \frac{\varepsilon}{2}. \tag{13}$$

2. Now, put

$$A_\varepsilon = \bigcup_{j=1}^{N} E_j$$

and prove that $\mu^*(A \triangle A_\varepsilon) < \varepsilon$.

2*a*. Since

$$A \subset \bigcup_{j=1}^{\infty} E_j,$$

one has

$$A \setminus A_\varepsilon \subset \bigcup_{j=1}^{\infty} E_j \setminus A_\varepsilon.$$

Since

$$A_\varepsilon \subset \bigcup_{j=1}^{\infty} E_j,$$

one can use the monotonicity and subtractivity of $\tilde{\mu}$. Together with estimate (13), this gives

$$\tilde{\mu}(A \setminus A_\varepsilon) \leq \tilde{\mu}(\bigcup_{j=1}^{\infty} E_j \setminus A_\varepsilon) < \frac{\varepsilon}{2}.$$

2*b*. The inclusion

$$A_\varepsilon \setminus A \subset \bigcup_{j=1}^{\infty} E_j \setminus A$$

implies

$$\tilde{\mu}(A_\varepsilon \setminus A) \leqslant \tilde{\mu}(\bigcup_{j=1}^{\infty} E_j \setminus A) = \tilde{\mu}(\bigcup_{j=1}^{\infty} E_j) - \tilde{\mu}(A) < \frac{\varepsilon}{2}.$$

Here we used the same properties of $\tilde{\mu}$ as above and the choice of the cover (E_j).

3. Finally,

$$\tilde{\mu}(A \triangle A_\varepsilon) \leqslant \tilde{\mu}(A \setminus A_\varepsilon) + \tilde{\mu}(A_\varepsilon \setminus A).$$

■

4 Monotone Classes and Uniqueness of Extension of Measure

Definition 4.1 *A collection of sets,* $\mathfrak{M}$ *is called* a monotone class *if together with any monotone sequence of sets* $\mathfrak{M}$ *contains the limit of this sequence.*

Example. Any σ-ring. (This follows from the Exercise 1. below).

Exercises.

1. Prove that any σ-ring is a monotone class.
2. If a ring is a monotone class, then it is a σ-ring.

We shall denote by $\mathfrak{M}(\mathfrak{K})$ the minimal monotone class containing $\mathfrak{K}$.

Theorem 4.1 *Let* $\mathfrak{K}$ *be a ring of sets,* $\mathfrak{K}_\sigma$ *the* σ*-ring generated by* $\mathfrak{K}$*. Then* $\mathfrak{M}(\mathfrak{K}) = \mathfrak{K}_\sigma$*.*

Proof. 1. Clearly, $\mathfrak{M}(\mathfrak{K}) \subset \mathfrak{K}_\sigma$. Now, it suffices to prove that $\mathfrak{M}(\mathfrak{K})$ is a ring. This follows from the Exercise (2) above and from the minimality of $\mathfrak{K}_\sigma$.

2. $\mathfrak{M}(\mathfrak{K})$ <u>is a ring.</u>

2a. For $B \subset X$, set

$$\mathfrak{K}_B = \{A \subset X : A \cup B, A \cap B, A \setminus B, B \setminus A \in \mathfrak{M}(\mathfrak{K})\}.$$

This definition is symmetric with respect to A and B, therefore $A \in \mathfrak{K}_B$ implies $B \in \mathfrak{K}_A$.

2b. $\mathfrak{K}_B$ <u>is a monotone class.</u>

Let $(A_j) \subset \mathfrak{K}_B$ be a monotonically increasing sequence. Prove that the union, $A = \bigcup A_j$ belongs to $\mathfrak{K}_B$.

Since $A_j \in \mathfrak{K}_B$, one has $A_j \cup B \in \mathfrak{K}_B$, and so

$$A \cup B = \bigcup_{j=1}^{\infty}(A_j \cup B) \in \mathfrak{M}(\mathfrak{K}).$$

In the same way,

$$A \setminus B = (\bigcup_{j=1}^{\infty} A_j) \setminus B = \bigcup_{j=1}^{\infty}(A_j \setminus B) \in \mathfrak{M}(\mathfrak{K});$$

$$B \setminus A = B \setminus (\bigcup_{j=1}^{\infty} A_j) = \bigcap_{j=1}^{\infty}(B \setminus A_j) \in \mathfrak{M}(\mathfrak{K}).$$

Similar proof is for the case of decreasing sequence (A_j).

2c. If $B \in \mathfrak{K}$ then $\mathfrak{M}(\mathfrak{K}) \subset \mathfrak{K}_B$.

Obviously, $\mathfrak{K} \subset \mathfrak{K}_B$. Together with minimality of $\mathfrak{M}(\mathfrak{K})$, this implies $\mathfrak{M}(\mathfrak{K}) \subset \mathfrak{K}_B$.

2d. If $B \in \mathfrak{M}(\mathfrak{K})$ then $\mathfrak{M}(\mathfrak{K}) \subset \mathfrak{K}_B$.

Let $A \in \mathfrak{K}$. Then $\mathfrak{M}(\mathfrak{K}) \subset \mathfrak{K}_A$. Thus if $B \in \mathfrak{M}(\mathfrak{K})$, one has $B \in \mathfrak{K}_A$, so $A \in \mathfrak{K}_B$.

Hence what we have proved is $\mathfrak{K} \subset \mathfrak{K}_B$. This implies $\mathfrak{M}(\mathfrak{K}) \subset \mathfrak{K}_B$.

2e. It follows from 2a. — 2d. that if $A, B \in \mathfrak{M}(\mathfrak{K})$ then $A \in \mathfrak{K}_B$ and so $A \cup B$, $A \cap B$, $A \setminus B$ and $B \setminus A$ all belong to $\mathfrak{M}(\mathfrak{K})$. ■

Theorem 4.2 *Let $\mathfrak{A}$ be an algebra of sets, μ and ν two measures defined on the σ-algebra $\mathfrak{A}_\sigma$ generated by $\mathfrak{A}$. Then $\mu \restriction \mathfrak{A} = \nu \restriction \mathfrak{A}$ implies $\mu = \nu$.*

Proof. Choose $A \in \mathfrak{A}_\sigma$, then $A = \lim_{n\to\infty} A_n, \quad A_n \in \mathfrak{A}$, for $\mathfrak{A}_\sigma = \mathfrak{M}(\mathfrak{A})$. Using continuity of measure, one has

$$\mu(A) = \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \nu(A_n) = \nu(A).$$

■

Theorem 4.3 *Let $\mathfrak{A}$ be an algebra of sets, $B \subset X$ such that for any $\varepsilon > 0$ there exists $A_\varepsilon \in \mathfrak{A}$ with $\mu^*(B \triangle A_\varepsilon) < \varepsilon$. Then $B \in \tilde{\mathfrak{A}}$.*

Proof. 1. Since any outer measure is semi-additive, it suffices to prove that for any $E \subset X$ one has

$$\mu^*(E) \geqslant \mu^*(E \cap B) + \mu^*(E \cap B^c).$$

2a. Since $\mathfrak{A} \subset \tilde{\mathfrak{A}}$, one has

$$\mu^*(E \cap A_\varepsilon) + \mu^*(E \cap A_\varepsilon^c) \leqslant \mu^*(E). \tag{14}$$

2b. Since $A \subset B \cup (A \triangle B)$ and since the outer measure μ^* is monotone and semi-additive, there is an estimate $|\mu^*(A) - \mu^*(B)| \leqslant \mu^*(A \triangle B)$ for any $A, B \subset X$. (*C.f.* the proof of similar fact for measures above).

2c. It follows from the monotonicity of μ^* that

$$|\mu^*(E \cap A_\varepsilon) - \mu^*(E \cap B)| \leqslant \mu^*((E \cap A_\varepsilon) \triangle (E \cap B)) \leqslant \mu(A_\varepsilon \cap B) < \varepsilon.$$

Therefore, $\mu^*(E \cap A_\varepsilon) > \mu^*(E \cap B) - \varepsilon$.

In the same manner, $\mu^*(E \cap A_\varepsilon^c) > \mu^*(E \cap B^c) - \varepsilon$.

$2d$. Using (14), one obtains

$$\mu^*(E) > \mu^*(E \cap B) + \mu^*(E \cap B^c) - 2\varepsilon.$$

■

5 The Lebesgue Measure on the real line $\mathbb{R}^1$

5.1 The Lebesgue Measure of Bounded Sets of $\mathbb{R}^1$

Put $\mathfrak{A}$ for the algebra of all finite unions of semi-segments (semi-intervals) on $\mathbb{R}^1$, i.e. all sets of the form

$$A = \bigcup_{j=1}^{k} [a_j, b_j).$$

Define a mapping $\mu : \mathfrak{A} \longrightarrow \mathbb{R}$ by:

$$\mu(A) = \sum_{j=1}^{k} (b_j - a_j).$$

Theorem 5.1 *μ is a measure.*

Proof. 1. All properties including the (finite) additivity are obvious. The only thing to be proved is the σ-additivity.

Let $(A_j) \subset \mathfrak{A}$ be such a countable disjoint family that

$$A = \bigsqcup_{j=1}^{\infty} A_j \in \mathfrak{A}.$$

The condition $A \in \mathfrak{A}$ means that $\bigsqcup A_j$ is *a finite* union of intervals.

2. For any positive integer n,

$$\bigcup_{j=1}^{n} A_j \subset A,$$

hence

$$\sum_{j=1}^{n} \mu(A_j) \leqslant \mu(A),$$

and

$$\sum_{j=1}^{\infty} \mu(A_j) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(A_j) \leqslant \mu(A).$$

3. Now, let A^ε a set obtained from A by the following construction. Take a connected component of A. It is a semi-segment of the form $[s, t)$. Shift slightly on the left its right-hand end, to obtain a (closed) segment. Do it with all components of A, in such a way that

$$\mu(A) < \mu(A^\varepsilon) + \varepsilon. \tag{15}$$

Apply a similar procedure to each semi-segment shifting their left end point to the left $A_j = [a_j, b_j)$, and obtain (open) intervals, A_j^ε with

$$\mu(A_j^\varepsilon) < \mu(A_j) + \frac{\varepsilon}{2^j}. \tag{16}$$

4. By the construction, A^ε is a compact set and (A_j^ε) its open cover. Hence, there exists a positive integer n such that

$$\bigcup_{j=1}^{n} A_j^\varepsilon \supset A^\varepsilon.$$

Thus

$$\mu(A^\varepsilon) \leqslant \sum_{j=1}^{n} \mu(A_j^\varepsilon).$$

The formulas (15) and (16) imply

$$\mu(A) < \sum_{j=1}^{n} \mu(A_j^\varepsilon) + \varepsilon \leqslant \sum_{j=1}^{n} \mu(A_j) + \sum_{j=1}^{n} \frac{\varepsilon}{2^j} + \varepsilon,$$

thus

$$\mu(A) < \sum_{j=1}^{\infty} \mu(A_j) + 2\varepsilon.$$

■

Now, one can apply the Carathèodory's scheme developed above, and obtain the measure space $(\tilde{\mathfrak{A}}, \tilde{\mu})$. The result of this extension is called *the Lebesgue measure.* We shall denote the Lebesgue measure on $\mathbb{R}^1$ by m.

Exercises.

1. A one point set is measurable, and its Lebesgue measure is equal to 0.
2. The same for a countable subset in $\mathbb{R}^1$. In particular, $m(\mathbb{Q} \cap [0, 1]) = 0$.
3. Any open or closed set in $\mathbb{R}^1$ is Lebesgue measurable.

Definition 5.1 Borel algebra *of sets,* $\mathfrak{B}$ *on the real line* $\mathbb{R}^1$ *is a* σ*-algebra generated by all open sets on* $\mathbb{R}^1$*. Any element of* $\mathfrak{B}$ *is called* a Borel set.

Exercise. Any Borel set is Lebesgue measurable.

Theorem 5.2 *Let* $E \subset \mathbb{R}^1$ *be a Lebesgue measurable set. Then for any* $\varepsilon > 0$ *there exists an open set* $G \supset E$ *such that* $m(G \setminus E) < \varepsilon$*.*

Proof. Since E is measurable, $m^*(E) = m(E)$. According the definition of an outer measure, for any $\varepsilon > 0$ there exists a cover $A = \bigcup [a_k, b_k) \supset E$ such that

$$m(A) < m(E) + \frac{\varepsilon}{2}.$$

Now, put

$$G = \bigcup (a_k - \frac{\varepsilon}{2^{k+1}}, b^k).$$

■

Problem. Let $E \subset \mathbb{R}^1$ be a bounded Lebesgue measurable set. Then for any $\varepsilon > 0$ there exists a compact set $F \subset E$ such that $m(E \setminus F) < \varepsilon$. (*Hint:* Cover E with a semi-segment and apply the above theorem to the σ-algebra of measurable subsets in this semi-segment).

Corollary. *For any $\varepsilon > 0$ there exist an open set G and a compact set F such that $G \supset E \supset F$ and $m(G \setminus F) < \varepsilon$.*

Such measures are called *regular.*

5.2 The Lebesgue Measure on the Real Line $\mathbb{R}^1$

We now abolish the condition of boundness.

Definition 5.2 *A set A on the real numbers line $\mathbb{R}^1$ is Lebesgue measurable if for any positive integer n the bounded set $A \cap [-n, n)$ is a Lebesgue measurable set.*

Definition 5.3 *The Lebesgue measure on $\mathbb{R}^1$ is*

$$m(A) = \lim_{n \to \infty} m(A \cap [-n, n)).$$

Definition 5.4 *A measure is called σ-finite if any measurable set can be represented as a countable union of subsets each has a finite measure.*

Thus the Lebesgue measure m is σ-finite.

Problem. The Lebesgue measure on $\mathbb{R}^1$ is regular.

5.3 The Lebesgue Measure in $\mathbb{R}^d$

Definition 5.5 *We call* a d-dimensional rectangle *in $\mathbb{R}^d$ any set of the form*

$$\{x : x \in \mathbb{R}^d : a_i \leqslant x_i < b_i\}.$$

Using rectangles, one can construct the Lebesque measure in $\mathbb{R}^d$ in the same fashion as we deed for the $\mathbb{R}^1$ case.

6 Measurable functions

Let X be a set, $\mathfrak{A}$ a σ-algebra on X.

Definition 6.1 *A pair* $(X, \mathfrak{A})$ *is called a* measurable space.

Definition 6.2 *Let* f *be a function defined on a measurable space* $(X, \mathfrak{A})$*, with values in the extended real number system. The function* f *is called* measurable *if the set*

$$\{x : f(x) > a\}$$

is measurable for every real a*.*

Example.

Theorem 6.1 *The following conditions are equivalent*

$$\{x : f(x) > a\} \text{ is measurable for every real } a. \tag{17}$$
$$\{x : f(x) \geq a\} \text{ is measurable for every real } a. \tag{18}$$
$$\{x : f(x) < a\} \text{ is measurable for every real } a. \tag{19}$$
$$\{x : f(x) \leq a\} \text{ is measurable for every real } a. \tag{20}$$

Proof. The statement follows from the equalities

$$\{x : f(x) \geq a\} = \bigcap_{n=1}^{\infty} \{x : f(x) > a - \frac{1}{n}\}, \tag{21}$$
$$\{x : f(x) < a\} = X \setminus \{x : f(x) \geq a\}, \tag{22}$$
$$\{x : f(x) \leq a\} = \bigcap_{n=1}^{\infty} \{x : f(x) < a + \frac{1}{n}\}, \tag{23}$$
$$\{x : f(x) > a\} = X \setminus \{x : f(x) \leq a\} \tag{24}$$

Theorem 6.2 *Let* (f_n) *be a sequence of measurable functions. For* $x \in X$ *set*

$$g(x) = \sup_n f_n(x) (n \in \mathbb{N})$$

$$h(x) = \limsup_{n \to \infty} f_n(x).$$

Then g *and* h *are measurable.*

Proof.

$$\{x : g(x) \leq a\} = \bigcap_{n=1}^{\infty} \{x : f_n(x) \leq a\}.$$

Since the LHS is measurable it follows that the RHS is measurable too. The same proof works for inf.

Now

$$h(x) = \inf g_m(x),$$

where

$$g_m(x) = \sup_{n \geq m} f_n(x).$$

Theorem 6.3 *Let f and g be measurable real-valued functions defined on X. Let F be real and continuous function on $\mathbb{R}^2$. Put*

$$h(x) = F(f(x), g(x)) \ (x \in X).$$

Then h is measurable.

Proof. Let $G_a = \{(u, v) : F(u, v) > a\}$. Then G_a is an open subset of $\mathbb{R}^2$, and thus

$$G_a = \bigcup_{n=1}^{\infty} I_n$$

where (I_n) is a sequence of open intervals

$$I_n = \{(u, v) : a_n < u < b_n, c_n < v < d_n\}.$$

The set $\{x : a_n < f(x) < b_n\}$ is measurable and so is the set

$$\{x : \ (f(x), g(x)) \in I_n\} = \{x : \ a_n < f(x) < b_n\} \cap \{x : \ c_n < g(x) < d_n\}.$$

Hence the same is true for

$$\{x : \ h(x) > a\} = \{x : \ (f(x), g(x)) \in G_a\} = \bigcup_{n=1}^{\infty} \{x : (f(x), g(x)) \in I_n\}.$$

Corollories. Let f and g be measurable. Then the following functions are measurable

$$(i) f + g \tag{25}$$

$$(ii) f \cdot g \tag{26}$$

$$(iii) |f| \tag{27}$$

$$(iv) \frac{f}{g} (\text{if} g \neq 0) \tag{28}$$

$$(v) \max\{f, g\}, \min\{f, g\} \tag{29}$$

(30)

since $\max\{f, g\} = 1/2(f + g + |f - g|), \ \min\{f, g\} = 1/2(f + g - |f - g|)$.

6.1 Step functions (simple functions)

Definition 6.3 *A real valued function $f : X \to \mathbb{R}$ is called simple function if it takes only a finite number of distinct values.*

We will use below the following notation

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$$

Theorem 6.4 *A simple function $f = \sum_{j=1}^{n} c_j \chi_{E_j}$ is measurable if and only if all the sets E_j are measurable.*

Exercise. Prove the theorem.

Theorem 6.5 *Let f be real valued. There exists a sequence (f_n) of simple functions such that $f_n(x) \longrightarrow f(x)$ as $n \to \infty$, for every $x \in X$. If f is measurable, (f_n) may be chosen to be a sequence of measurable functions. If $f \geq 0$, (f_n) may be chosen monotonically increasing.*

Proof. If $f \geq 0$ set

$f_n(x) = \sum_{i=1}^{n \cdot 2^n} \frac{i-1}{2^n} \chi_{E_{n_i}} + n\chi_{F_n}$

where

$E_{n_i} = \{x : \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n}\},\ F_n = \{x : f(x) \geq n\}.$

The sequence (f_n) is monotonically increasing, f_n is a simple function. If $f(x) < \infty$ then $f(x) < n$ for a sufficiently large n and $|f_n(x) - f(x)| < 1/2^n$. Therefore $f_n(x) \longrightarrow f(x)$. If $f(x) = +\infty$ then $f_n(x) = n$ and again $f_n(x) \longrightarrow f(x)$.

In the general case $f = f^+ - f^-$, where

$f^+(x) := \max\{f(x), 0\},\ f^-(x) := -\min\{f(x), 0\}.$

Note that if f is bounded then $f_n \longrightarrow f$ uniformly.

7 Integration

Definition 7.1 *A triple $(X, \mathfrak{A}, \mu)$, where $\mathfrak{A}$ is a σ-algebra of subsets of X and μ is a measure on it, is called a measure space.*

Let $(X, \mathfrak{A}, \mu)$ be a measure space. Let $f : X \mapsto \mathbb{R}$ be a simple measurable function.

$$f(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(x) \tag{31}$$

and

$$\bigcup_{i=1}^{n} E_i = X, \;\; E_i \cap E_j = \varnothing \;(i \neq j).$$

There are different representations of f by means of (31). Let us choose the representation such that all c_i are distinct.

Definition 7.2 *Define the quantity*

$$I(f) = \sum_{i=1}^{n} c_i \mu(E_i).$$

First, we derive some properties of $I(f)$.

Theorem 7.1 *Let f be a simple measurable function. If $X = \bigsqcup_{j=1}^{k} F_j$ and f takes the constant value b_j on F_j then*

$$I(f) = \sum_{j=1}^{k} b_j \mu(F_j).$$

Proof. Clearly, $E_i = \bigsqcup_{j:\; b_j = c_i} F_j$.

$$\sum_i c_i \mu(E_i) = \sum_{i=1}^{n} c_i \mu(\bigsqcup_{j:\; b_j = c_i} F_j) = \sum_{i=1}^{n} c_i \sum_{j:\; b_j = c_i} \mu(F_j) = \sum_{j=1}^{k} b_j \mu(F_j).$$

■

This show that the quantity $I(f)$ is well defined.

Theorem 7.2 *If f and g are measurable simple functions then*

$$I(\alpha f+\beta g)=\alpha I(f)+\beta I(g).$$

Proof. Let $f(x)=\sum_{j=1}^{n} b_j\chi_{F_j}(x),\ X=\bigsqcup_{j=1}^{n} F_j$, $g(x)=\sum_{k=1}^{m} c_k\chi_{G_k}(x),\ X=\bigsqcup_{k=1}^{n} G_k$.

Then

$$\alpha f+\beta g=\sum_{j=1}^{n}\sum_{k=1}^{m}(\alpha b_j+\beta c_k)\chi_{E_{jk}}(x)$$

where $E_{jk}=F_j\cap G_k$.

Exercise. Complete the proof.

Theorem 7.3 *Let f and g be simple measurable functions. Suppose that $f\le g$ everywhere except for a set of measure zero. Then*

$$I(f)\le I(g).$$

Proof. If $f\le g$ everywhere then in the notation of the previous proof $b_j\le c_k$ on E_{jk} and $I(f)\le I(g)$ follows.

Otherwise we can assume that $f\le g+\phi$ where ϕ is non-negative measurable simple function which is zero every exept for a set N of measure zero. Then $I(\phi)=0$ and

$$I(f)\le I(g+\phi)=I(f)+I(\phi)=I(g).$$

Definition 7.3 *If $f:X\mapsto\mathbb{R}^1$ is a non-negative measurable function, we define the Lebesgue integral of f by*

$$\int f d\mu := \sup I(\phi)$$

where sup *is taken over the set of all simple functions ϕ such that $\phi\le f$.*

Theorem 7.4 *If f is a simple measurable function then $\int f d\mu=I(f)$.*

Proof. Since $f\le f$ it follows that $\int f d\mu\ge I(f)$.

On the other hand, if $\phi\le f$ then $I(\phi)\le I(f)$ and also

$$\sup_{\phi\le f} I(\phi)\le I(f)$$

which leads to the inequality

$$\int f d\mu\le I(f).$$

■

Definition 7.4 ***1.*** *If A is a measurable subset of X $(A \in \mathfrak{A})$and f is a non-negative measurable function then we define*

$$\int_A f d\mu = \int f \chi_A d\mu.$$

2.

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

if at least one of the terms in RHS is finite. If both are finite we call f integrable.

Remark. Finiteness of the integrals $\int f^+ d\mu$ and $\int f^- d\mu$ is equivalent to the finitenes of the integral

$$\int |f| d\mu.$$

If it is the case we write $f \in L^1(X, \mu)$ or simply $f \in L^1$ if there is no ambiguity.

The following properties of the Lebesgue integral are simple consequences of the definition. The proofs are left to the reader.

- If f is measurable and bounded on A and $\mu(A) < \infty$ then f is integrable on A.
- If $a \le f(x) \le b$ $(x \in A)),\ \mu(A) < \infty$ then

$$a\mu(A) \le \int_A f d\mu \le b\mu(a).$$

- If $f(x) \le g(x)$ for all $x \in A$ then

$$\int_A f d\mu \le \int_A g d\mu.$$

- Prove that if $\mu(A) = 0$ and f is measurable then

$$\int_A f d\mu = 0.$$

The next theorem expresses an important property of the Lebesgue integral. As a consequence we obtain convergence theorems which give the main advantage of the Lebesgue approach to integration in comparison with Riemann integration.

Theorem 7.5 *Let f be measurable on X. For $A \in \mathfrak{A}$ define*

$$\phi(A) = \int_A f d\mu.$$

Then ϕ is countably additive on $\mathfrak{A}$.

Proof. It is enough to consider the case $f \geq 0$. The general case follows from the decomposition $f = f^+ - f^-$.

If $f = \chi_E$ for some $E \in \mathfrak{A}$ then

$$\mu(A \cap E) = \int_A \chi_E d\mu$$

and σ-additivity of ϕ is the same as this property of μ.

Let $f(x) = \sum_{k=1}^n c_k \chi_{E_k}(x)$, $\bigsqcup_{k=1}^n E_k = X$. Then for $A = \bigsqcup_{i=1}^\infty A_i$, $A_i \in \mathfrak{A}$ we have

$$\begin{aligned} \phi(A) = \int_A f d\mu = \int f \chi_A d\mu = \sum_{k=1}^n c_k \mu(E_k \cap A) \\ = \sum_{k=1}^n c_k \mu(E_k \cap (\bigsqcup_{i=1}^\infty A_i)) = \sum_{k=1}^n c_k \mu(\bigsqcup_{i=1}^\infty (E_k \cap A_i)) \\ = \sum_{k=1}^n c_k \sum_{i=1}^\infty \mu(E_k \cap A_i) = \sum_{i=1}^\infty \sum_{k=1}^n c_k \mu(E_k \cap A_i) \\ \text{(the series of positive numbers)} \\ = \sum_{i=1}^\infty \int_{A_i} f d\mu = \sum_{i=1}^\infty \phi(A_i). \end{aligned}$$

Now consider general positive f's. Let φ be a simple measurable function and $\varphi \leq f$. Then

$$\int_A \varphi d\mu = \sum_{i=1}^\infty \int_{A_i} \varphi d\mu \leq \sum_{i=1}^\infty \phi(A_i).$$

Therefore the same inequality holds for sup, hence

$$\phi(A) \leq \sum_{i=1}^\infty \phi(A_i).$$

Now if for some i $\phi(A_i) = +\infty$ then $\phi(A) = +\infty$ since $\phi(A) \geq \phi(A_n)$. So assume that $\phi(A_i) < \infty \forall i$. Given $\varepsilon > 0$ choose a measurable simple function φ such that $\varphi \leq f$ and

$$\int_{A_1} \varphi d\mu \geq \int_{A_1} f d\mu - \varepsilon, \quad \int_{A_2} \varphi d\mu \geq \int_{A_2} f - \varepsilon.$$

Hence

$$\phi(A_1 \cup A_2) \geq \int_{A_1 \cup A_2} \varphi d\mu = \int_{A_1} + \int_{A_2} \varphi d\mu \geq \phi(A_1) + \phi(A_2) - 2\varepsilon,$$

so that $\phi(A_1 \cup A_2) \geq \phi(A_1) + \phi(A_2)$.

By induction

$$\phi(\bigcup_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} \phi(A_i).$$

Since $A \supset \bigcup_{i=1}^{n} A_i$ we have that

$$\phi(A) \geq \sum_{i=1}^{n} \phi(A_i).$$

Passing to the limit $n \to \infty$ in the RHS we obtain

$$\phi(A) \geq \sum_{i=1}^{\infty} \phi(A_i).$$

This completes the proof.■

Corollary. If $A \in \mathfrak{A},\ B \subset A$ and $\mu(A \setminus B) = 0$ then

$$\int_A f d\mu = \int_B f d\mu.$$

Proof.

$$\int_A f d\mu = \int_B f d\mu + \int_{A \setminus B} f d\mu = \int_B f d\mu + 0.$$

■

Definition 7.5 *f and g are called equivalent* ($f \sim g$ *in writing) if* $\mu(\{x : \ f(x) \neq g(x)\}) = 0$.

It is not hard to see that $f \sim g$ is relation of equivalence.
(i) $f \sim f$, (ii) $f \sim g,\ g \sim h \Rightarrow f \sim h$, (iii) $f \sim g \Leftrightarrow g \sim f$.

Theorem 7.6 *If* $f \in L^1$ *then* $|f| \in L^1$ *and*

$$\left| \int_A f d\mu \right| \leq \int_A |f| d\mu$$

Proof.

$$-|f| \leq f \leq |f|$$

Theorem 7.7 *(Monotone Convergence Theorem)*
Let (f_n) be nondecreasing sequence of nonnegative measurable functions with limit f. Then

$$\int_A f d\mu = \lim_{n\to\infty} \int_A f_n d\mu, \ \ A \in \mathfrak{A}$$

Proof. First, note that $f_n(x) \leq f(x)$ so that

$$\lim_n \int_A f_n d\mu \leq \int f d\mu$$

It is remained to prove the opposite inequality.
For this it is enough to show that for any simple φ such that $0 \leq \varphi \leq f$ the following inequality holds

$$\int_A \varphi d\mu \leq \lim_n \int_A f_n d\mu$$

Take $0 < c < 1$. Define

$$A_n = \{x \in A : f_n(x) \geq c\varphi(x)\}$$

then $A_n \subset A_{n+1}$ and $A = \bigcup_{n=1}^{\infty} A_n$.
Now observe

$$c\int_A \varphi d\mu = \int_A c\varphi d\mu = \lim_{n\to\infty} \int_{A_n} c\varphi d\mu \leq$$

(this is a consequence of σ-additivity of ϕ proved above)

$$\leq \lim_{n\to\infty} \int_{A_n} f_n d\mu \leq \lim_{n\to\infty} \int_A f_n d\mu$$

Pass to the limit $c \to 1$.■

Theorem 7.8 *Let $f = f_1 + f_2, f_1, f_2 \in L^1(\mu)$. Then $f \in L^1(\mu)$ and*

$$\int f d\mu = \int f_1 d\mu + \int f_2 d\mu$$

Proof. First, let $f_1, f_2 \geq 0$. If they are simple then the result is trivial. Otherwise, choose monotonically increasing sequences $(\varphi_{n,1}), (\varphi_{n,2})$ such that $\varphi_{n,1} \to f_1$ and $\varphi_{n,2} \to f_2$.

Then for $\varphi_n = \varphi_{n,1} + \varphi_{n,2}$

$$\int \varphi_n d\mu = \int \varphi_{n,1} d\mu + \int \varphi_{n,2} d\mu$$

and the result follows from the previous theorem.

If $f_1 \geq 0$ and $f_2 \leq 0$ put

$$A = \{x : \ f(x) \geq 0\}, \ B = \{x : \ f(x) < 0\}$$

Then f, f_1 and $-f_2$ are non-negative on A.

Hence $\int_A f_1 = \int_A f d\mu + \int_A (-f_2) d\mu$

Similarly

$$\int_B (-f_2) d\mu = \int_B f_1 d\mu + \int_B (-f) d\mu$$

The result follows from the additivity of integral. ■

Theorem 7.9 *Let $A \in \mathfrak{A}$, (f_n) be a sequence of non-negative measurable functions and*

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \ x \in A$$

Then

$$\int_A f d\mu = \sum_{n=1}^{\infty} \int_A f_n d\mu$$

Exercise. Prove the theorem.

Theorem 7.10 ***(Fatou's lemma)***
If (f_n) is a sequence of non-negative measurable functions defined a.e. and

$$f(x) = \underline{\lim}_{n\to\infty} f_n(x)$$

then

$$\int_A f d\mu \leq \underline{\lim}_{n\to\infty} \int_A f_n d\mu$$

$$A \in \mathfrak{A}$$

Proof. Put $\quad g_n(x) = \inf_{i\geq n} f_i(x)$
Then by definition of the lower limit $\lim_{n\to\infty} g_n(x) = f(x)$.
Moreover, $g_n \leq g_{n+1},\ g_n \leq f_n$. By the monotone convergence theorem

$$\int_A f d\mu = \lim_n \int_A g_n d\mu = \underline{\lim}_n \int_A g_n d\mu \leq \underline{\lim}_n \int_A f_n d\mu.$$

Theorem 7.11 ***(Lebesgue's dominated convergence theorem)***
Let $A \in \mathfrak{A}$, (f_n) be a sequence of measurable functions such that $f_n(x) \to f(x)$ $(x \in A.)$ Suppose there exists a function $g \in L^1(\mu)$ on A such that

$$|f_n(x)| \leq g(x)$$

Then

$$\lim_n \int_A f_n d\mu = \int_A f d\mu$$

Proof. From $|f_n(x)| \leq g(x)$ it follows that $f_n \in L^1(\mu)$. Sinnce $f_n + g \geq 0$ and $f + g \geq 0$, by Fatou's lemma it follows

$$\int_A (f+g) d\mu \leq \underline{\lim}_n \int_A (f_n + g)$$

or

$$\int_A f d\mu \leq \underline{\lim}_n \int_A f_n d\mu.$$

Since $g - f_n \geq 0$ we have similarly

$$\int_A (g-f) d\mu \leq \underline{\lim}_n \int_A (g - f_n) d\mu$$

so that

$$-\int_A f d\mu \leq -\underline{\lim}_n \int_A f_n d\mu$$

which is the same as

$$\int_A f d\mu \geq \overline{\lim}_n \int_A f_n d\mu$$

This proves that

$$\underline{\lim}_n \int_A f_n d\mu = \overline{\lim}_n \int_A f_n d\mu = \int_A f d\mu.$$

8 Comparison of the Riemann and the Lebesgue integral

To distinguish we denote the Riemann integral by $(R)\int_a^b f(x)dx$ and the Lebesgue integral by $(L)\int_a^b f(x)dx$.

Theorem 8.1 *If a finction f is Riemann integrable on $[a,b]$ then it is also Lebesgue integrable on $[a,b]$ and*

$$(L)\int_a^b f(x)dx = (R)\int_a^b f(x)dx.$$

Proof. Boundedness of a function is a necessary condition of being Riemann integrable. On the other hand, every bounded measurable function is Lebesgue integarble. So it is enough to prove that if a function f is Riemann integrable then it is measurable.

Consider a partition π_m of $[a,b]$ on $n = 2^m$ equal parts by points $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ and set

$$\underline{f}_m(x) = \sum_{k=0}^{2^m-1} m_k \chi_k(x), \ \overline{f}_m(x) = \sum_{k=0}^{2^m-1} M_k \chi_k(x),$$

where χ_k is a charactersitic function of $[x_k, x_{k+1})$ clearly,

$$\underline{f}_1(x) \le \underline{f}_2(x) \le \ldots \le f(x),$$

$$\overline{f}_1(x) \ge \overline{f}_2(x) \ge \ldots \ge f(x).$$

Therefore the limits

$$\underline{f}(x) = \lim_{m\to\infty} \underline{f}_m(x), \ \ \overline{f}(x) = \lim_{m\to\infty} \overline{f}_m(x)$$

exist and are measurable. Note that $\underline{f}(x) \le f(x) \le \overline{f}(x)$. Since $\underline{f}_m$ and $\overline{f}_m$ are simple measurable functions, we have

$$(L)\int_a^b \underline{f}_m(x)dx \le (L)\int_a^b \underline{f}(x)dx \le (L)\int_a^b \overline{f}(x)dx \le (L)\int_a^b \overline{f}_m(x)dx.$$

Moreover,

$$(L)\int_a^b \underline{f}_m(x)dx = \sum_{k=0}^{2^m-1} m_k \Delta x_k = \underline{s}(f, \pi_m)$$

and similarly

$$(L)\int_a^b \overline{f}_m(x) = \overline{s}(f, \pi_m).$$

So

$$\underline{s}(f, \pi_m) \leq (L)\int_a^b \underline{f}(x)dx \leq (L)\int_a^b \overline{f}(x)dx \leq \overline{s}(f, \pi_m).$$

Since f is Riemann integrable,

$$\lim_{m\to\infty} \underline{s}(f, \pi_m) = \lim_{m\to\infty} \overline{s}(f, \pi_m) = (R)\int_a^b f(x)dx.$$

Therefore

$$(L)\int_a^b (\overline{f}(x) - \underline{f}(x))dx = 0$$

and since $\overline{f} \geq \underline{f}$ we conclude that

$$f = \overline{f} = \underline{f} \quad \text{almost everywhere.}$$

From this measurability of f follows. ■

9 L^p-spaces

Let $(X, \mathfrak{A}, \mu)$ be a measure space. In this section we study $L^p(X, \mathfrak{A}, \mu)$-spaces which occur frequently in analysis.

9.1 Auxiliary facts

Lemma 9.1 *Let p and q be real numbers such that $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1$ (this numbers are called conjugate). Then for any $a > 0,\ b > 0$ the inequality*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

holds.

Proof. Note that $\varphi(t) := \frac{t^p}{p} + \frac{1}{q} - t$ with $t \geq 0$ has the only minimum at $t = 1$. It follows that

$$t \leq \frac{t^p}{p} + \frac{1}{q}.$$

Then letting $t = ab^{-\frac{1}{p-1}}$ we obtain

$$\frac{a^p b^{-q}}{p} + \frac{1}{q} \geq ab^{-\frac{1}{p-1}},$$

and the result follows.■

Lemma 9.2 *Let $p \geq 1,\ a, b \in \mathbb{R}$. Then the inequality*

$$|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p).$$

holds.

Proof. For $p = 1$ the statement is obvious. For $p > 1$ the function $y = x^p,\ x \geq 0$ is convex since $y'' \geq 0$. Therefore

$$\left(\frac{|a| + |b|}{2}\right)^p \leq \frac{|a|^p + |b|^p}{2}.■$$

9.2 The spaces L^p, $1 \leq p < \infty$. Definition

Recall that two measurable functions are said to be equaivalent (with respect to the measure μ) if they are equal μ-a;most everywhere.

The space $L^p = L^p(X, \mathfrak{A}, \mu)$ consists of all μ-equaivalence classes of $\mathfrak{A}$-measurable functions f such that $|f|^p$ has finite integral over X with respect to μ.

We set

$$\|f\|_p := \left(\int_X |f|^p d\mu \right)^{1/p}.$$

9.3 Hölder's inequality

Theorem 9.3 *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Let f and g be measurable functions, $|f|^p$ and $|g|^q$ be integrable. Then fg is integrable andthe inequality*

$$\int_X |fg| d\mu \leq \left(\int_X |f|^p d\mu \right)^{1/p} \left(\int_X |g|^q d\mu \right)^{1/q}.$$

Proof. It suffices to consider the case

$$\|f\|_p > 0, \ \|g\|_q > 0.$$

Let

$$a = |f(x)| \|f\|_p^{-1}, \ b = |g(x)| \|g\|_q^{-1}.$$

By Lemma 1

$$\frac{|f(x)g(x)|}{\|f\|_p \|g\|_q} \leq \frac{|f(x)|^p}{p\|f\|_p^p} + \frac{|g(x)|^q}{q\|g\|_q^q}.$$

After integration we obtain

$$\|f\|_p^{-1} \|g\|_q^{-1} \int_X |fg| d\mu \leq \frac{1}{p} + \frac{1}{q} = 1. \qquad \blacksquare$$

9.4 Minkowski's inequality

Theorem 9.4 *If $f, g \in L^p$, $p \geq 1$, then $f + g \in L^p$ and*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof. If $\|f\|_p$ and $\|g\|_p$ are finite then by Lemma 2 $|f+g|^p$is integrable and $\|f+g\|_p$ is well-defined.

$$|f(x)+g(x)|^p = |f(x)+g(x)||f(x)+g(x)|^{p-1} \leq |f(x)||f(x)+g(x)|^{p-1}+|g(x)||f(x)+g(x)|^{p-1}.$$

Integratin the last inequality and using Hölder's inequality we obtain

$$\int_X |f+g|^p d\mu \leq \left(\int_X |f|^p d\mu\right)^{1/p} \left(\int_X |f+g|^{(p-1)q} d\mu\right)^{1/q} + \left(\int_X |g|^p d\mu\right)^{1/p} \left(\int_X |f+g|^{(p-1)q} d\mu\right)^{1/q}.$$

The result follows. ■

9.5 $L^p,\ 1 \leq p < \infty$, is a Banach space

It is readily seen from the properties of an integral and Theorem 9.3 that $L^p,\ 1 \leq p < \infty$, is a vector space. We introduced the quantity $\|f\|_p$. Let us show that it defines a norm on $L^p,\ 1 \leq p < \infty$,. Indeed,

1. By the definition $\|f\|_p \geq 0$.
2. $\|f\|_p = 0 \Longrightarrow f(x) = 0$ for μ-almost all $x \in X$. Since L^p consists of μ-eqivalence classes, it follows that $f \sim 0$.
3. Obviously, $\|\alpha f\|_p = |\alpha|\|f\|_p$.
4. From Minkowski's inequality it follows that $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

So $L^p,\ 1 \leq p < \infty$, is a normed space.

Theorem 9.5 $L^p,\ 1 \leq p < \infty$, *is a Banach space.*

Proof. It remains to prove the completeness.

Let (f_n) be a Cauchy sequence in L^p. Then there exists a subsequence $(f_{n_k})(k \in \mathbb{N})$ with n_k increasing such that

$$\|f_m - f_{n_k}\|_p < \frac{1}{2^k} \quad \forall m \geq n_k.$$

Then

$$\sum_{i=1}^{k} \|f_{n_{i+1}} - f_{n_i}\|_p < 1.$$

Let

$$g_k := |f_{n_1}| + |f_{n_2} - f_{n_1}| + \ldots + |f_{n_{k+1}} - f_{n_k}|.$$

Then g_k is monotonocally increasing. Using Minkowski's inequality we have

$$\|g_k^p\|_1 = \|g_k\|_p^p \leq \left(\|f_{n_1}\|_p + \sum_{i=1}^{k} \|f_{n_{i+1}} - f_{n_i}\|_p \right)^p < (\|f_{n_1}\|_p + 1)^p.$$

Put

$$g(x) := \lim_k g_k(x).$$

By the monotone convergence theorem

$$\lim_k \int_X g_k^p d\mu = \int_A g^p d\mu.$$

Moreover, the limit is finite since $\|g_k^p\|_1 \leq C = (\|f_{n_1}\|_p + 1)^p$.

Therefore

$$|f_{n_1}| + \sum_{j=1}^{\infty} |f_{n_{j+1}} - f_{n_j}| \quad \text{converges almost everywhere}$$

and so does

$$f_{n_1} + \sum_{j=1}^{\infty} (f_{n_{j+1}} - f_{n_j}),$$

which means that

$$f_{n_1} + \sum_{j=1}^{N} (f_{n_{j+1}} - f_{n_j}) = f_{n_{N+1}} \quad \text{converges almost everywhere as } N \to \infty.$$

Define

$$f(x) := \lim_{k \to \infty} f_{n_k}(x)$$

where the limit exists and zero on the complement. So f is measurable.

Let $\epsilon > 0$ be such that for $n, m > N$

$$\|f_n - f_m\|_p^p = \int_X |f_n - f_m|^p d\mu < \epsilon/2.$$

Then by Fatou's lemma

$$\int_X |f - f_m|^p d\mu = \int_X \lim_k |f_{n_k} - f_m|^p d\mu \leq \underline{\lim}_k \int_X |f_{n_k} - f_m|^p d\mu$$

which is less than ϵ for $m > N$. This proves that

$$\|f - f_m\|_p \to 0 \text{ as } m \to \infty. \qquad \blacksquare$$