Acyclic Colorings of Products of Cycles

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Abstract

An *acyclic coloring* of a graph G is a proper coloring of the vertex set of G such that G contains no bichromatic cycles. The *acyclic chromatic number* of a graph G is the minimum number k such that G has an acyclic coloring with k colors. In this paper, acyclic colorings of products of paths and cycles are considered. We determine the acyclic chromatic numbers of three such products: grid graphs, cylinders, and toroids.

1 Introduction

A *k-coloring* of a graph G with vertex set $V(G)$ is a labeling $f : V(G) \to \{1, \ldots, k\}$. A k-coloring of a graph is a *proper coloring* provided any two adjacent vertices have distinct colors. An *acyclic coloring* of a graph G is a proper coloring of G such that the subgraph of G induced by any two color classes of G contains no cycles. The chromatic number of G, denoted $\chi(G)$, is the minimum k such that G has a proper k-coloring; the *acyclic chromatic number* of a graph G, denoted $AC(G)$, is the minimum number k such that G has an acyclic k-coloring.

Acyclic colorings were introduced by Grünbaum in [7] where he showed that a graph with maximum degree 3 has an acyclic 4-coloring. In [5], Burnstein proved that a graph with maximum degree 4 has an acyclic 5-coloring. In this paper, we determine the acyclic chromatic numbers of certain graphs with maximum degree 4. In particular, we find the acyclic chromatic numbers of products of paths and cycles. This is an extension of the work of Ferrin, Godard, and Raspaud [6] where acyclic colorings of certain grids (products of paths) are studied and of the authors and Villalpando [8] where acyclic colorings of products of trees are considered. Additional references on acyclic colorings include papers by Berman

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and Albertson [1]; Borodin [4]; Alon, Mohar, and Sanders [3]; Alon, McDiarmid, and Reed [2]; Mohar [9]; Skulrattanakulchai [12]; and Nowakowski and Rall [11].

The product we are taking is the usual *Cartesian* (or *box*) product. The vertex set of $G \Box H$ is the Cartesian product $V(G) \times V(H)$ of the vertex sets of G and H. There is an edge between two vertices (a, b) and (x, y) of the product if and only if they are adjacent in exactly one coordinate and agree in the other.

Acyclic colorings are hereditary in the sense that the restriction of an acyclic coloring to a subgraph is an acyclic coloring. Thus, the acyclic chromatic number is nondecreasing from subgraph to supergraph.

All of the graphs we consider will be simple (no loops or multiple edges). As usual, P_m denotes the *m-path*, the path on m vertices; and C_n denotes the *n-cycle*, the cycle on n vertices. All paths P_m considered have $m \geq 2$. The product of two paths is a *grid graph*. The product of a path and a cycle is a *cylinder*. The product of two cycles is a *toroid*. Cylinders will generally be written as $P_m \Box C_n$ and toroids as $C_m \Box C_n$.

In [8], it was shown that the acyclic chromatic number of the product of two (nondegenerate) trees is 3. The grid graphs are a special case. When wrap-around is introduced, the situation becomes more complex. In this paper, we focus on determining the acyclic chomatic numbers of cyclinders and toroids. Our results are summarized in the table below.

type of graph	graph	acyclic chromatic number	reference
grid graph	$P_m \Box P_n$	3	[6], [8]
cylinder	$P_m \Box C_n$	3 if $n \neq 4$	Theorem 4.5
		4 if $n = 4$	Theorem 4.6
toroid	$C_m \Box C_n$	4 if $(m, n) \neq (3, 3)$	Theorem 3.1
	$C_3 \Box C_3$	5	Theorem 3.2

Notation. Suppose φ is a k-coloring of a grid $P_m \Box P_n$, cylinder $P_m \Box C_n$, or toroid $C_m \Box C_n$. Let $c(i, j)$ denote the color assigned to the vertex (u_i, v_j) in the product. Then the colors $c(i, j)$ can be viewed as the entries of an m by n matrix M, called the *color matrix* of φ. The first row and column of the color matrix are called the *axes*.

For any pair of colors a and b, let $C_{a,b}$ denote the set of all vertices colored either a or b. Being an acyclic coloring means $C_{a,b}$ induces a forest for all pairs of colors a and b. The classes $C_{a,b}$ are called *bichromate classes*.

2 A lower bound on the acyclic chromatic number

We begin with a simple yet very useful lower bound on the acyclic chromatic number of an arbitrary graph. This is a refinement of [6, Theorem 1] suggested by Mohar [10].

Theorem 2.1. *The acyclic chromatic number of a graph G satisfies*

$$AC(G) > \frac{|E(G)|}{|V(G)|} + 1.$$

Proof. Let G be a graph with $|V(G)| = n$ and $|E(G)| = m$. Suppose that φ is an acyclic coloring of G with k colors. Denote the color classes of φ by $C_1, \ldots, C_k$, and set $|C_i| = n_i$. Since G has no bichromatic cycles, the subgraph of G induced by any two color classes C_i and C_j is a forest T_{ij}. Clearly, $|E(T_{ij})| \leq n_i + n_j - 1$. As a result,

$$m = \sum_{1 \leq i < j \leq k} |E(T_{ij})| \leq \sum_{1 \leq i < j \leq k} n_i + n_j - 1 = (k-1) \sum_{i=1}^{k} n_i - \binom{k}{2} = (k-1)n - \binom{k}{2}.$$

Then $\frac{m}{n} \leq (k-1) - \frac{\binom{k}{2}}{n}$ and so

$$k \geq \frac{m}{n} + 1 + \frac{\binom{k}{2}}{n}.$$

Since $\frac{\binom{k}{2}}{n} > 0$, we conclude that

$$AC(G) > \frac{m}{n} + 1.$$

∎

Corollary 2.2. *Let $G_1, \ldots, G_d$ be graphs. The acyclic chromatic number of the product $G_1 \square \cdots \square G_d$ satisfies*

$$AC(G_1 \square \cdots \square G_d) > \sum_{i=1}^{d} \frac{|E(G_i)|}{|V(G_i)|} + 1.$$

Proof. Let $G = G_1 \square \cdots \square G_d$. Suppose that $V(G_1) = \{u_1, \ldots, u_n\}$. Then $(u_1, v_2, \ldots, v_d) \in V(G)$ is adjacent to $\deg u_1 |V(G_2)| \cdots |V(G_d)|$ vertices of the form $(u_j, v_2, \ldots, v_d)$. Thus, there are

$$\begin{aligned} \tfrac{1}{2}(\deg u_1 + \cdots + \deg u_n)|V(G_2)| \cdots |V(G_d)| &= \tfrac{1}{2}\left(2|E(G_1)|\right)|V(G_2)| \cdots |V(G_d)| \\ &= |E(G_1)||V(G_2)| \ldots |V(G_d)| \end{aligned}$$

edges of G of the form $\{(u_1, v_2, \ldots, v_d), (u_j, v_2, \ldots, v_d)\}$ in G. Continuing in this manner, we see that

$$|E(G)| = \left(\prod_{j=1}^{d} |V(G_j)| \right) \sum_{i=1}^{d} \frac{|E(G_i)|}{|V(G_i)|}.$$

By Theorem 2.1,

$$AC(G) > \frac{|E(G)|}{|V(G)|} + 1 = \sum_{i=1}^{d} \frac{|E(G_i)|}{|V(G_i)|} + 1.$$

∎

As an immediate consequence, we obtain the following lower bounds on the acyclic chromatic numbers of a cylinder and a toroid.

Corollary 2.3. *The acyclic chromatic numbers of a cylinder and a toroid satisfy*

$$AC(P_m \square C_n) \geq 3 \quad \text{and} \quad AC(C_m \square C_n) \geq 4.$$

Using the fact that cylinders and toroids both have maximum degree 4 and the main result of [5], we conclude that

$$3 \leq AC(P_m \square C_n) \leq 5 \quad \text{and} \quad 4 \leq AC(C_m \square C_n) \leq 5.$$

Exact values for these quantities will be found in Sections 3 and 4.

3 Toroids

3.1 Toroids $C_m \square C_n$ have acyclic 4-colorings unless $m = n = 3$

In this subsection, we give an acyclic 4-coloring of a toroid $C_m \square C_n$ where $(m, n) \neq (3, 3)$. To prove that this coloring is indeed acyclic, we will use the notion of a regular vertex. A vertex v is called *regular* if any two neighbors of v of the same color correspond to entries in the same row or column of the color matrix; otherwise, v is said to be *irregular*.

Theorem 3.1. *If $(m, n) \neq (3, 3)$, then the acyclic chromatic number of the toroid $C_m \square C_n$ is $AC(C_m \square C_n) = 4$.*

Proof. According to Corollary 2.3, $AC(C_m \square C_n) \geq 4$. Let M denote the $(m-2) \times (n-1)$ matrix with alternating rows

$$M = \begin{bmatrix} 1 & 3 & 1 & 3 & 1 & 3 & \cdots \\ 0 & 2 & 0 & 2 & 0 & 2 & \cdots \\ 1 & 3 & 1 & 3 & 1 & 3 & \cdots \\ 0 & 2 & 0 & 2 & 0 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

We will use M to give a color matrix for $C_m \square C_n$. There are three cases to consider, based on the parities of m and n. The color matrix of each is as follows:

Case 1: m even, n odd

$$\left[\begin{array}{ccccccccc|c} \multicolumn{9}{c|}{} & 2 \\ \multicolumn{9}{c|}{} & 1 \\ \multicolumn{9}{c|}{} & 2 \\ \multicolumn{9}{c|}{M} & 1 \\ \multicolumn{9}{c|}{} & \vdots \\ \multicolumn{9}{c|}{} & 2 \\ \multicolumn{9}{c|}{} & 1 \\ \hline 1 & 3 & 1 & 3 & \cdots & 1 & 3 & 1 & 0 & 3 \\ 2 & 0 & 2 & 0 & \cdots & 2 & 0 & 2 & 1 & 0 \end{array}\right]$$

Case 2: m even, n even

$$\left[\begin{array}{ccccccccc|c} \multicolumn{9}{c|}{} & 2 \\ \multicolumn{9}{c|}{} & 3 \\ \multicolumn{9}{c|}{} & 2 \\ \multicolumn{9}{c|}{M} & 3 \\ \multicolumn{9}{c|}{} & \vdots \\ \multicolumn{9}{c|}{} & 2 \\ \multicolumn{9}{c|}{} & 1 \\ \hline 1 & 3 & 1 & 3 & \cdots & 1 & 3 & 2 & & 3 \\ 2 & 0 & 2 & 0 & \cdots & 2 & 0 & 3 & & 0 \end{array}\right]$$

Case 3: m odd, n odd, and $(m, n) \neq (3, 3)$

$$\left[\begin{array}{ccccccccc|c} \multicolumn{9}{c|}{} & 0 \\ \multicolumn{9}{c|}{} & 3 \\ \multicolumn{9}{c|}{} & 0 \\ \multicolumn{9}{c|}{M} & 3 \\ \multicolumn{9}{c|}{} & \vdots \\ \multicolumn{9}{c|}{} & 3 \\ \multicolumn{9}{c|}{} & 0 \\ \hline 0 & 2 & 0 & 2 & \cdots & 0 & 2 & 0 & 1 & 3 \\ 3 & 0 & 3 & 0 & \cdots & 3 & 0 & 3 & 2 & 1 \end{array}\right]$$

Note that each color matrix defines a proper coloring of the toroid $C_m \square C_n$. We claim that this coloring is acyclic. Suppose that a regular vertex is part of a bichromatic cycle. Then, by definition of regular, that cycle must be a row or a column. However, there are no bichromatic rows or columns. As a result, each vertex of a bichromatic cycle must irregular. It is easy to check that there are no bichromatic cycles formed using only irregular vertices.
■

Table 1: Canonical color matrices for $C_3 \Box C_3$.

$$M_1 = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \\ ? & ? & ? \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & 1 \\ 2 & 3 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & 1 \\ 1 & ? & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

Table 2: Bichromatic cycles for the canonical colorings of $C_3 \Box C_3$.

Matrix	Class	Bichromatic Cycle
M_1	$C_{1,2}$	$(1,2) - (1,3) - (2,3) - (2,2) - (1,2)$
M_2	$C_{0,2}$	$(1,1) - (1,3) - (3,3) - (3,1) - (1,1)$
M_3	$C_{1,3}$	$(2,1) - (2,3) - (3,3) - (3,1) - (2,1)$
M_4	$C_{0,1}$	$(1,1) - (1,2) - (2,2) - (2,3) - (3,3) - (3,1) - (1,1)$

3.2 An acyclic coloring of $C_3 \Box C_3$

Next, we address the case where $(m, n) = (3, 3)$.

Theorem 3.2. *The acyclic chromatic number of the toroid $C_3 \Box C_3$ is $AC(C_3 \Box C_3) = 5$.*

Proof. The bulk of the proof is to show that every proper 4-coloring of $G = C_3 \Box C_3$ is equivalent, via an automorphism of $C_3 \Box C_3$, to one of the colorings shown in Table 1. Table 2 shows a bichromatic cycle for each of these four canonical colorings. Note that any permutation of the rows and/or columns of the color matrix corresponds to an automorphism of $C_3 \Box C_3$ because of the completeness of the factors.

The color matrix of a proper 4-coloring has the following property:

(*) Each entry differs from its four neighbors vertically and horizontally, including wrap-around.

In particular, the entries in each row are different. Since there are 4 colors used, it cannot be that all rows have the same 3 colors. Hence a pair of consecutive rows have different color sets. Let these be rows one and two. Rotate the columns so that the color in position (2,1) is different from the three colors in row one. Thus the color matrix appears in the form of A below where w, x, y, z are different and where ? denotes an unknown entry.

$$\overset{\text{A}}{\begin{bmatrix} x & y & z \\ w & ? & ? \\ ? & ? & ? \end{bmatrix}}, \quad \overset{\text{B}}{\begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & ? \\ ? & ? & ? \end{bmatrix}}, \quad \overset{\text{C}}{\begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & 1 \\ \{1,2\} & \{2,3\} & \{0,3\} \end{bmatrix}}, \quad \overset{M_5}{\begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}}.$$

There a two possibilities at this point: either x appears in the second row or it does not. If x is not in the second row, then the coloring is of type M_1 shown in Table 1. If x does appear in the second row, then the colors can be chosen and the last two columns can be permuted if necessary to obtain the matrix B. Now $c(2,2) = 0$ implies $c(2,3) = 1$ by (*). Hence we have the first two rows of matrix C. The last row shows the two color options available for that entry according to (*). By (*), these options lead to four possible colorings as shown below. The matrix type is shown with each:

$$[2,3,0] \to (M_2); \quad [1,3,0] \to (M_3); \quad [1,2,0] \to (M_3); \quad [1,2,3] \to (M_4)$$

Therefore, G does not have an acyclic 4-coloring. By [5], G has an acyclic 5-coloring as G is 4-valent. It is also easy to see that color matrix M_5 above defines an explicit acyclic 5-coloring of G. ∎

Notice that since $C_3 \square C_3$ is vertex-transitive, the coloring in M_5 shows that $C_3 \square C_3$ is *AC-critical*, meaning that the removal of any vertex reduces the acyclic chromatic number.

3.3 A nearly acyclic 3-coloring

To conclude this section, we consider a 3-coloring of certain toroids which is nearly acyclic. Consider $G = C_m \square C_n$ where $\gcd(m,n) = 3$ and at least one of m or n is greater than 3. According to Corollary 2.3, G does not have an acyclic 3-coloring. However, G almost has such a coloring. We will demonstrate this now.

Define a coloring of G by assigning the color $\varphi(i,j) := i + j \bmod 3$ to the vertex in position (i,j). The coloring φ is a proper coloring since $3|m$ and $3|n$. Through any vertex there are exactly 2 bichromatic "zig-zag paths". To see that each closes and is actually a cycle, note that

$$\varphi(1,j) = j + 1 = \varphi(m, j+1) \text{ and } \varphi(i,n) = i = \varphi(i-1, 1)$$

for $1 \le i \le m$ and $1 \le j \le n$. Now suppose such a cycle contains h horizontal edges and v vertical edges. Since horizontal and vertical edges alternate, $h = v$. Because the cycle closes horizontally, $m|h$; because the cycle closes vertically, $m|v$. It follows that $m|h$ and $n|h$, and so $\operatorname{lcm}(m,n)|h$. Note that $\operatorname{lcm}(m,n) = \frac{mn}{\gcd(m,n)} = \frac{mn}{3}$. This forces $h + v = 2h = \frac{2}{3}mnt$ for some integer t. Since G has mn vertices, $t = 1$ and so $h + v = \frac{2}{3}mn$. Hence each of these cycles contains $\frac{2}{3}$ of the vertices of G. Therefore, there is exactly one bichromatic cycle for each pair of colors.

Finally, choose any two vertices v_1 and v_2 which are of different colors and are not adjacent. The removal of v_1 and v_2 leaves a graph with an acyclic 3-coloring. Alternatively, recoloring these two vertices gives an acyclic 4-coloring of G. Hence, G almost has an acyclic 3-coloring.

4 Cylinders

4.1 Conformal colorings

A k-coloring φ is said to be *conformal* if and only if

$$c(s,t) \equiv c(s,1) + c(1,t) - c(1,1) \mod k. \tag{1}$$

for all $1 \le s \le m$ and $0 \le t \le n$. Note that a conformal coloring is completely determined by the colors on the axes of the color matrix. In this section, we show that any proper 3-coloring of a grid graph, cylinder, or toroid without bichromatic 4-cycles is conformal. To do this, the following uniformity result is useful.

Lemma 4.1. *Suppose that φ is a proper 3-coloring of a grid graph $P_m \Box P_n$ or cylinder $P_m \Box C_n$ with no bichromatic 4-cycles. Let $1 \le i \le i+s \le m$ and $1 \le j \le j+t \le n$. Then*

$$c(i+s, j+t) - c(i, j+t) \equiv c(i+s, j) - c(i,j) \bmod 3.$$

Proof. We proceed by double induction on s and t. If $s = t = 1$, then the three colors used lie on a 4-cycle, and all three available colors must be used. There are four cases as shown below, each of which satisfies the desired condition.

a	a+1
a+1	a+2

a	a+2
a+1	a

a	a+1
a+2	a

a	a+2
a+2	a+1

We now establish the case $s = 1$ for all t. We have, modulo 3, the following equalities:

$$c(i+1, j+t) - c(i, j+t) \equiv c(i+1, j+t-1) - c(i, j+t-1) \equiv c(i+1, j) - c(i,j).$$

The first equality follows from the base case $s = t = 1$; the other follows by induction. Hence, for all t, $c(i+1, j+t) - c(i, j+t) \equiv c(i+1, j) - c(i,j) \bmod 3$.

It remains to establish the equality for all s. Suppose that

$$c(i+s-1, j+t) - c(i, j+t) \equiv c(i+s-1, j) - c(i,j) \bmod 3;$$

that is, suppose $c(i+s-1, j+t) - c(i+s-1, j) \equiv c(i, j+t) - c(i,j) \bmod 3$. The result follows immediately since $c(i+1, j+t) - c(i+1, j) \equiv c(i, j+t) - c(i,j) \bmod 3$. ∎

Theorem 4.2. *Suppose that φ is a proper 3-coloring of a grid graph or cylinder with no bichromatic 4-cycles. Then the color matrix of φ is determined by the entries on its axes. In fact, φ is conformal.*

Proof. Taking $i = j = 1$ in Lemma 4.1 shows that

$$c(s,t) \equiv c(s,1) + c(1,t) - c(1,1) \bmod 3.$$

∎

Remark 4.3. *Lemma 4.1 and Theorem 4.2 also hold for a toroid $C_m \Box C_n$.*

4.2 Cylinders have acyclic 3-colorings if $n \neq 4$

Consider the cylinder $P_m \square C_n$ where $n \neq 4$. We wish to describe an acyclic 3-coloring φ of $P_m \square C_n$. According to Theorem 4.2, to define such a coloring, we only need to give the colors of the axes; the formula in Theorem 4.2 can be used to extend the coloring on the axes to the full color matrix.

The strategy is the following. First, we specify a coloring φ via an appropriate coloring of the axes. We then show that the last row of the color matrix cannot be involved in any bichromatic cycle. The last row can then be removed and m reduced. We proceed again to show that the new last row is not involved in any bichromatic cycle. In this way, we show by induction that $P_m \square C_n$ contains no bichromatic cycles and φ is acyclic.

To facilitate the reduction argument, we say that a position in the last row *standard* if and only if it has color a and the distribution of colors in the six positions indicated around it are as shown below. The designations a, b, c indicate distinct colors.

	c	b	
c	b	**a**	c

or

	b	c	
c	**a**	b	c

In the course of an argument about bichromatic cycles, an element is called *excluded* provided it has already been shown that that vertex cannot lie in a bichromatic cycle. A vertex that is not excluded is *usable*. The status of vertices will shift from usable to excluded.

The next result is an observation concerning usable and excluded vertices.

Lemma 4.4. *1. If a vertex v colored a is usable, then at least two usable neighbors of v must have the same color b different from a.*

2. A standard vertex is excluded.

Proof. 1) If v is on a bichromatic cycle, the color of its two neighbors on the cycle are such a color. If not, v can be excluded.

2) Suppose that v is a standard vertex with color a. Then, by definition, v is on the last row and has a neighbor w on the last row with color b, and v and w have all their neighbors shown in the diagrams. In each of the two cases, v has exactly two neighbors colored b. Thus if v is on a bichromatic cycle, the other color on the cycle must be b. Hence, w must be on the cycle. It follows that w must have at least two neighbors colored a, which it does not. Hence v is not usable and so is excluded. ■

Theorem 4.5. *If $n \neq 4$, then $AC(P_m \square C_n) = 3$.*

Proof. According to Theorem 4.2, to prescribe an acyclic 3-coloring of the cylinder $P_m \square C_n$, it suffices to assign colors to the axes; i.e., it suffices to specify the first row and the first column of the color matrix M. Use the cyclic sequence $012012 \ldots 012$ to color the first column of M. Since there is no vertical wrap-around, this sequence can stop on 0, 1 or 2 (so there is no parity condition on m). To specify the first row of the color matrix, we distinguish three cases based on the residue of n mod 3.

Case 1. $n \equiv 0 \bmod 3$.

Color first row $012012\ldots012012$. Because the coloring is cyclic horizontally, all positions on the bottom row are standard. By Lemma 4.4, each vertex on the bottom row is excluded from bichromatic cycles.

Case 2. $n \equiv 1 \bmod 3$.

Since $n \neq 4$, the case hypothesis implies $n \geq 7$. Color the first row $012012\ldots0120121$. There is now a hill triad 121 at the end and a valley triad 101 over the wrap-around.

For (m, j) to be standard, we must have $3 \leq j \leq m - 2$. In particular, $(m, 3)$ and $(m, n-2)$ are both standard and hence are excluded from bichromatic cycles.

The following table shows the bottom four rows of the color matrix. The double vertical line indicates where the wrap-around occurs. We may assume that the last row starts with 2. In fact, it could start with 1 or 0, but these other alternatives (which depend on m mod 3) are all equivalent via a cyclic shift in the colors to the case that $c(m, 1) = 2$. Thus, the bichromate classes are the same. The standard positions are marked by x.

$n \equiv 1 \bmod 3$

2	0	1	$\cdots$	2	0	1	0	2	0	1
0	1	2	$\cdots$	0	1	$\underline{2}$	1	0	1	2
1	2	0	$\cdots$	1	$\underline{2}$	$\underline{0}$	$\underline{2}$	1	2	0
2	0	x	$\cdots$	x	x	1	0	2	0	x

Consider the 1 in position $(m, n-1)$. Its only two usable neighbors are both colored 0, and neither is adjacent to another 1. Hence this 1 cannot be used in a bichromatic cycle and hence is excluded.

Now consider the 0 in position (m, n). Its only two usable neighbors are both colored 2. However, if we trace the pathway determined by the 2 in position $(m-1, n)$, we see that it terminates in deadends (shown by underlining). Hence the 0 in (m, n) cannot be used, so it is excluded.

Now the 2 in position $(m, 1)$ has only two potentially usuable neighbors (since the 0 in the (m, n) position is excluded) and they are of opposite colors. Hence the 2 in position $(m, 1)$ is excluded. At this point, the 0 in the $(m, 2)$ position has only one usable neighbor, and it is excluded. Therefore, the entire last row is excluded from bichromatic cycles.

Note that the standard positions play a crucial role in the argument by ruling out certain options for viable neighbors initially. If this coloring is tried when $n = 4$, it is easy to find bichromatic 6-cycles in any two consecutive rows. (cf. [8]).

Case 3. $n \equiv 2 \bmod 3$.

Color the first row $012012\ldots01201$.

For (m, j) to be standard, we must have $3 \leq j \leq n-1$. In particular, $(m, 3)$ and $(m, n-1)$ are both standard and hence cannot be used in a bichromatic cycle.

As in the previous case, the following table shows the bottom four rows of the color matrix with the standard positions marked by an x.

$n \equiv 2 \bmod 3$

2	0	1	$\cdots$	2	0	1	2	0	2	0	1
0	1	2	$\cdots$	0	1	2	0	1	0	1	2
1	2	0	$\cdots$	1	2	0	1	2	1	2	0
2	0	x	$\cdots$	x	x	x	x	0	2	0	x

The 0 in position (m, n) has only 2's as usuable neighbors. However, the 2 above has only one 0 as neighbor, so it cannot lie in a 0,2-bichromatic cycle. Hence position (m, n) is excluded.

The 2 in position $(m, 1)$ has only two usable neighbors and they have different colors. Hence $(m, 1)$ is also excluded. This leaves the 0 in position $(m, 2)$ with only one potentially usable neighbor (above), so it cannot be in a bichromatic cycle either. Thus the entire bottom row has been shown to not lie in any bichromatic cycle.

In each of the three cases, the last row cannot be involved in any bichromatic cycles. By induction, we see that the specified colorings are indeed acyclic. As a result, $AC(P_m \Box C_n) = 3$ provided $n \neq 4$. ■

4.3 Cylinders with $n = 4$

We conclude this section by giving the acyclic chromatic number of cylinders which have C_4 as a factor.

Corollary 4.6. *The acyclic chromatic number of a cylinder* $P_m \Box C_4$ *is* $AC(P_m \Box C_4) = 4$.

Proof. In [8], it is shown that $AC(P_m \Box C_4) \geq 4$. Now Theorem 3.1 gives

$$AC(P_m \Box C_4) \leq AC(C_{m+1} \Box C_4) \leq 4$$

since $P_m \Box C_4$ is a subgraph of $C_{m+1} \Box C_4$. ■

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