DISTANCE COLORINGS OF HYPERCUBES FROM $\mathbb{Z}_2\mathbb{Z}_4$-LINEAR CODES

GRETCHEN L. MATTHEWS

ABSTRACT. In this paper, we give distance-ℓ colorings of the hypercube using nonlinear binary codes which are images of codes which exhibit linear structure over more general rings; specifically, we employ images of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. This is an extension of the work of Fu, Ling, and Xing (Discrete Appl. Math. 161 (2013)) where the authors demonstrate that distance-ℓ colorings may be obtained from nonlinear codes which are binary images of codes that have a linear structure over $\mathbb{Z}_4$.

1. Introduction

In this paper, we give distance-ℓ colorings of the hypercube using nonlinear binary codes which are images of codes which exhibit linear structure over more general rings; specifically, we employ images of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. A distance-ℓ coloring of a graph is a coloring of the vertices so that any two vertices at distance ℓ or less from one another receive different colors. These colorings were introduced in the context of the scalability of optical networks [14] and have been considered by many researchers (see, for instance, [11, 12, 13]). Binary linear codes are a standard tool to provide a distance-ℓ coloring of an n-dimensional hypercube, as the vertices of this graph are labeled with binary words of length n ([15, 16]). Although a nonlinear binary code may have better parameters than comparable binary ones (for instance, those length and same minimum distance can have many more codewords), nonlinear codes lack the structure that lends itself so nicely to the hypercube coloring problem. Recently, Fu et. al. [6] demonstrated that distance-ℓ colorings may be obtained from nonlinear codes which are binary images of codes that have a linear structure over $\mathbb{Z}_4$. In this paper, we consider extensions of their work.

This paper is organized as follows. Section 2 contains background information on distance-ℓ colorings, in particular those coming from binary linear codes. Section 3 details colorings from codes which are nonlinear as binary codes but are images of subgroups of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ and provides examples illustrating our results. Closing comments are found in Section 4.

2. Preliminaries

A binary code C of length n is a subset of $\mathbb{F}_2^n$, where $\mathbb{F}_2 := \{0,1\}$ is the finite field with two elements. Elements of C are called codewords. The Hamming distance between $w, w' \in \mathbb{F}_2^n$ is $d(w,w') := |\{i : w_i \neq w'_i\}|$. Note that $d(w,w') = wt(w-w')$, the weight of the word $w-w'$, meaning its number of nonzero entries. The quality of a code C is often measured by ratios

Department of Mathematical Sciences, Clemson University, Clemson, SC 29634-0975
phone: 864-656-5239; email: gmatthe@clemson.edu
G. L. Matthews' work is supported in part by NSA MSP-111006 and NSF DMS-1403062.

of the following parameters: n, the length of the code C; $M := |C|$, the size of the code C; and $d := \min\{d(c,c') : c, c' \in C, c \neq c'\}$, the minimum distance of C. For instance, one might wish to fix the length and cardinality (meaning the rate $\frac{M}{2^n}$) and attempt to maximize the minimum distance; alternatively, one might fix the length and minimum distance and attempt to maximize the cardinality of the code. A code C with these parameters is called an (n, M, d) code. A binary code C of length n is linear if and only if C is a subspace of $\mathbb{F}_2^n$.

Binary linear codes provide a known construction for a distance-ℓ colorings of hypercubes. A distance-ℓ coloring of a graph G with L colors is a labelling $f : V(G) \to \{1, \ldots, L\}$ of the vertex set $V(G)$ such that

$$f(u) \neq f(v) \text{ if } d(u,v) \leqslant \ell,$$

where $d(u,v) \leqslant \ell$ means that there is a path with at most ℓ edges containing vertices u and v. The distance-ℓ chromatic number of G is

$$\chi_\ell(G) := \min\{L : \exists \text{ distance-}\ell \text{ coloring of } G \text{ with } L \text{ colors}\}.$$

In this note, we restrict our attention to distance-ℓ colorings of hypercubes. Such colorings are considered in [11, 12, 13, 17, 18]. The n-dimensional hypercube H_n has $V(H_n) = \mathbb{F}_2^n$ as its set of vertices and edge set

$$E(H_n) = \{uv : u, v \in \mathbb{F}_2^n, d(u,v) = 1\}.$$

As is standard, we set

$$\chi_\ell(n) := \chi_\ell(H_n),$$

the distance-ℓ chromatic number of the n-dimensional hypercube H_n.

Let $C \subseteq \mathbb{F}_2^n$ be a binary linear $(n, 2^k, d)$ code. Consider the set of cosets of C in $\mathbb{F}_2^n$, $\mathbb{F}_2^n/C = \{u + C : u \in \mathbb{F}_2^n\}$, and label its elements $C_1, \ldots, C_{2^{n-k}}$. Notice that if $u+c, u+c' \in u+C$, then

$$d(u+c, u+c') = wt(u+c-(u+c')) = wt(c-c') \geqslant d,$$

because C is a linear code and $c, c' \in C$. Define a coloring of the vertices of the hypercube H_n by $f : V(H_n) \to \{1, \ldots, 2^{n-k}\}$ where

$$f(v) = i \text{ if and only if } v \in C_i.$$

Then f is a distance-$(d-1)$ coloring of H_n with 2^{n-k} colors. Hence,

$$\chi_{d-1}(n) \leqslant 2^{n-k}. \tag{1}$$

This demonstrates how one may use an $(n, 2^k, d)$ binary linear code to define a distance-$(d-1)$ coloring of the hypercube H_n.

Furthermore, given a distance-ℓ coloring f of the hypercube H_n with L colors and $i \in f(V(H_n)) = \{1, \ldots, L\}$, notice that $C_i := \{v \in \mathbb{F}_2^n : f(v) = i\}$ is a code of length n and minimum distance $\ell - 1$. In this way, we can see that a distance-ℓ coloring of the hypercube H_n gives rise to a partition $\mathbb{F}_2^n = C_1 \cup \cdots \cup C_L$ where each C_i is a code of length n and minimum distance at least $\ell - 1$. More on the connection between binary codes and colorings of hypercubes may be found in [13] where it is stated that the distance-ℓ chromatic number of the hypercube H_n is the "smallest number of binary codes with minimum distance $\ell + 1$ that form a partition" of $\mathbb{F}_2^n$.

A number of bounds on the distance-ℓ chromatic number have been specified, beginning with the works of [11, 12, 17]. These bounds are generalized and improved in [12, Theorem 1] to obtain

$$\sum_{i=0}^{t}\binom{n}{i}+\frac{\binom{n}{t}\left(\frac{n-t}{t+1}-\left\lfloor\frac{n-t}{t+1}\right\rfloor\right)}{\left\lfloor\frac{n}{t+1}\right\rfloor}\leqslant\chi_\ell\left(n\right)\leqslant 2^{\left\lfloor\log_2\sum_{i=0}^{\ell-1}\binom{n-1}{i}\right\rfloor+1} \tag{2}$$

if ℓ is even and

$$2\left(\sum_{i=0}^{t}\binom{n-1}{i}+\frac{\binom{n-1}{t}\left(\frac{n-1-t}{t+1}-\left\lfloor\frac{n-1-t}{t+1}\right\rfloor\right)}{\left\lfloor\frac{n-1}{t+1}\right\rfloor}\right)\leqslant\chi_\ell\left(n\right)\leqslant 2^{\left\lfloor\log_2\sum_{i=0}^{\ell-2}\binom{n-2}{i}\right\rfloor+2} \tag{3}$$

if ℓ is odd, where $t := \left\lfloor\frac{\ell}{2}\right\rfloor$.

3. Nonlinear codes and colorings

It has long been known that nonlinear codes can have better parameters than their comparable linear counterparts, meaning it is possible for a nonlinear code to have more codewords than any linear code of the same length and minimum distance. The most famous example is the Nordstrom-Robinson code which is a member of a family of nonlinear $(2^{r+1}, 2^{2^{r+1}-2r-2}, 6)$ binary codes, where $r \geqslant 3$ is odd. In [6], the authors exploit the $\mathbb{Z}_4$-linearity of these codes to give colorings of the hypercube. In this section, we generalize their construction and obtain further examples of distance-ℓ colorings of the hypercube from nonlinear binary codes which are images of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes.

We begin by detailing a distance-ℓ coloring using a $\mathbb{Z}_2\mathbb{Z}_4$-linear code. References on $\mathbb{Z}_2\mathbb{Z}_4$-linear codes include [2, 3]. Here, $\mathbb{Z}_2$ refers to the finite field $\mathbb{F}_2$ whereas $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ denotes the ring of integers modulo 4. We consider $\mathbb{Z}_4$ equipped with the Lee weight, which is given by $wt_L(0) = 0$, $wt_L(1) = 1$, $wt_L(2) = 2$, $wt_L(3) = 1$. Recall that the Gray map $\phi : \mathbb{Z}_4 \to \mathbb{Z}_2^2$ is given by

$$\begin{array}{ccc} 0 & \mapsto & 00 \\ 1 & \mapsto & 01 \\ 2 & \mapsto & 11 \\ 3 & \mapsto & 10 \end{array}$$

and extends to an isometry $(\mathbb{Z}_4^n, d_L) \to (\mathbb{Z}_2^{2n}, d)$ where d denotes the Hamming distance and $d_L(w, w') = \sum_{i=1}^{n} wt_L(w_i - w_i')$ for $w, w' \in \mathbb{Z}_4^n$.

Let $\mathcal{C}$ be a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, where α and β are non-negative integers. Define

$$\begin{array}{cccc} \Phi : & \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta & \to & \mathbb{Z}_2^n \\ & (x_1, \dots, x_\alpha, y_1, \dots, y_\beta) & \mapsto & (x_1, \dots, x_\alpha, \phi(y_1), \dots, \phi(y_\beta)) \end{array}$$

where $n = \alpha + 2\beta$ and ϕ denotes the Gray map as above. The Lee distance between $w, w' \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ is defined as

$$d_L\left(w, w'\right) := \sum_{i=1}^{\alpha} wt(w_i - w_i') + \sum_{i=\alpha+1}^{\alpha+\beta} wt_L(w_i - w_i').$$

Clearly, Φ gives an isometry $\left(\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta, d_L\right) \to (\mathbb{Z}_2^n, d)$. As a result, $\Phi(\mathcal{C})$ is a binary code of length n, minimum distance d, and $|\Phi(\mathcal{C})| = |\mathcal{C}|$. One may note that the minimum distance of $\Phi(\mathcal{C})$ is equal to that of $\mathcal{C}$, which is defined as $\min\{d_L(c,c') : c, c' \in \mathcal{C}, c \neq c'\}$.

Definition. A binary code C of length n is $\mathbb{Z}_2\mathbb{Z}_4$-linear if $C = \Phi(\mathcal{C})$ for some subgroup $\mathcal{C}$ of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ where $n = \alpha + 2\beta$. In this case, we say that $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code.

Notice that if $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, then $\mathcal{C} \cong \mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$ for some non-negative integers γ and δ. The code $\Phi(\mathcal{C})$ is then called a $\mathbb{Z}_2\mathbb{Z}_4$-linear code of type $(\alpha, \beta; \gamma, \delta)$.

The next result demonstrates that $\mathbb{Z}_2\mathbb{Z}_4$-linear codes may be used to provide distance-ℓ colorings of the hypercube.

Theorem 3.1. *If C is a $\mathbb{Z}_2\mathbb{Z}_4$-linear code of type $(\alpha, \beta; \gamma, \delta)$ and minimum distance d, then*

$$\chi_{d-1}(n) \leqslant 2^{n-(\gamma+2\delta)}$$

where $n = \alpha + 2\beta$.

Proof. Let C be a $\mathbb{Z}_2\mathbb{Z}_4$-linear code C of type $(\alpha, \beta; \gamma, \delta)$ and minimum distance d. Then there exists a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ such that $C = \Phi(\mathcal{C})$. Since $\mathcal{C} \cong \mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$, $|\mathcal{C}| = 2^{\gamma+2\delta}$.

Consider the coset space $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta / \mathcal{C}$, which has cardinality $\frac{2^n}{2^{\gamma+2\delta}} = 2^{n-(\gamma+2\delta)}$. Label the cosets of $\mathcal{C}$ in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ as $C_1, \ldots, C_{n-\gamma-2\delta}$, meaning

$$\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta / \mathcal{C} = \{C_1, \ldots, C_{n-\gamma-2\delta}\}.$$

Because $\{C_1, \ldots, C_{n-\gamma-2\delta}\}$ is a partition of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ and $\Phi : \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \to \mathbb{Z}_2^n$ is an isometry,

$$V(H_n) = \dot\cup_{i=1}^{n-\gamma-2\delta} \Phi(C_i);$$

that is, the images of the cosets partition the set of vertices of the hypercube. Define a coloring of the vertices of the hypercube H_n by

$$\begin{array}{rccl} f: & V(H_n) & \to & \{1, \ldots, 2^{n-(\gamma+2\delta)}\} \\ & v & \mapsto & i, \text{ where } v \in \Phi(C_i). \end{array}$$

We claim that f is a distance-$(d-1)$ coloring of H_n. To see this, suppose $f(v) = f(v')$. Then $v, v' \in \Phi(C_i)$ for some i, $1 \leqslant i \leqslant 2^{n-(\gamma+2\delta)}$. Hence, there exists $u \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ and $c, c' \in \mathcal{C}$ so that $v = \Phi(u + c)$ and $v' = \Phi(u + c')$. It follows that

$$d(v, v') = d(u + c, u + c') = wt((u + c) - (u + c')) = wt_L(c - c') \geqslant d.$$

Then f is a distance-$(d-1)$ coloring of H_n with $2^{n-(\gamma+2\delta)}$ colors. Hence,

$$\chi_{d_L-1}(n) \leqslant 2^{n-(\gamma+2\delta)}.$$

□

In the case that $\alpha = 0$, one may consider C as a $\mathbb{Z}_4$-linear code. Then, via Theorem 3.1, we recover a key result of Fu et. al. as a corollary.

Corollary 3.2. [6, Theorem 4] *If there exists a $\mathbb{Z}_4$-linear code C with parameters $(2n, 2^k, d)$, then C gives rise to a distance-$(d-1)$ coloring of H_{2n} and*

$$\chi_{d-1}(2n) \leqslant 2^{2n-k}.$$

Applying Corollary 3.2 to the $(2^r, 2^{2^{r+1}-2r-2}, 6)$ Preparata code over $\mathbb{Z}_4$ with $r \geqslant 3$ gives

$$\chi_5(2^{r+1}) \leqslant 2^{2r+2}.$$

In [6], they go on to show

$$\chi_5(2^{r+1}) = 2^{2r+2}.$$

Recently, Kiermaier and Zwanzger constructed a family of $\mathbb{Z}_4$-linear codes, called Teichmüller codes some of which have greater minimum distance than any linear code of equal length and cardinality. In particular, using Hjelmslev geometries and geometric dualization, they find $\mathbb{Z}_4$-linear codes C_r with parameters $(2^{2r+1} - 2^{r+1} + 2^{\frac{r-1}{2}}, 2^{2r+2}, 2^{2r} - 2^r)$, for all odd $r \geqslant 3$ [9, Theorem 5]. For $r = 3, 5$, C_r has greater minimum distance than any linear code of equal length and cardinality. In the next result, we use the codes C_r to find distance-l colorings of certain hypercubes for particular values of l.

Theorem 3.3. *For all odd $r \geqslant 3$ and $n = 2^{2r+1} - 2^{r+1} + 2^{\frac{r-1}{2}}$*

$$\chi_{2^{2r}-2^r-1}(n) \leqslant 2^{n-2r-2}.$$

Proof. This follows immediately from Theorem 3.1 using the $\mathbb{Z}_4$-linear codes C_r mentioned above. □

In the next example, we consider the first code in the family discovered by Kiermaier and Zwanzger, meaning C_r with $r = 3$.

Example 1. The code C_3 is a nonlinear binary $(114, 2^8, 56)$ code. Applying Theorem 3.3, we see that

$$\chi_{55}(114) \leqslant 2^{106}.$$

As a basis for comparison, consider that (3) gives

$$2.46 \times 10^{26} \leqslant \chi_{55}(114) \leqslant 2^{112} \leqslant 5.19 \times 10^{33}.$$

Thus, Theorem 3.3 offers an improvement to the upper bound on $\chi_{55}(114)$; one may note that $2^{106} \sim 8.113 \times 10^{31}$.

Example 2. In [10], a geometric construction based on a hyperoval in the projective Hjelmslev plane over $\mathbb{Z}_4$ yields a nonlinear binary $(58, 2^7, 28)$ code, which has greater minimum distance than any binary linear code of the same length with the same number of codewords [9]. Applying Theorem 3.1, we see

$$\chi_{27}(58) \leqslant 2^{51},$$

whereas (3) gives

$$2^{42.6629} \leqslant \chi_{27}(58) \leqslant 2^{56}.$$

Similar examples can be found to give

$$\chi_{27}(60) \leqslant 2^{52}$$

which might be compared with

$$2^{43.4} \leqslant \chi_{27}(60) \leqslant 2^{57}$$

as given by (3).

Nonlinear binary codes with parameters better than comparable binary linear codes are not limited to those found in [8, 9, 10]. In the next example, we utilize one which was originally found computationally.

Example 3. In this example, we consider a distance-11 coloring of the 64-dimensional hypercube H_{64}. Calderbank and McGuire found a $\mathbb{Z}_4$-linear code C with parameters $(64, 2^{37}, 12)$ [4]; the best known linear code of length 64 and dimension 37 has minimum distance 10 [7]. Applying Theorem 3.1, we see that

$$\chi_{11}(64) \leqslant 2^{27}.$$

As in the previous examples, we compare this with the bounds in [12, Theorem 1] which give $2 \times 10^7 \leqslant \chi_{11}(64) \leqslant 2^{36}$.

Remark 3.4. The code C in Example 3 is part of an infinite family of binary codes of lengths 2^{m+1} with $2^{2^{m+1}-5m-2}$ codewords, where m is odd [5]. For $m \neq 5$, the minumum distance of each code in this family equal to 8. Applying Theorem 3.1, we obtain

$$\chi_7\left(2^{m+1}\right) \leqslant 2^{5m+2}.$$

However, this bound is not so effective as [15, Theorem 15] gives

$$\chi_7\left(2^{m+1}\right) \leqslant 2^{3m+2}.$$

Each of the examples above utilizes a code which is $\mathbb{Z}_4$-linear, meaning that it is $\mathbb{Z}_2\mathbb{Z}_4$-linear of type $(0, \beta; 0, \delta)$. In the following example, we apply codes which are $\mathbb{Z}_2\mathbb{Z}_4$-linear of type $(\alpha, \beta; \gamma, \delta)$ with $\alpha, \gamma > 0$.

Example 4. In [1], $\mathbb{Z}_2\mathbb{Z}_4$-cyclic codes are considered. A number of such codes are described explicitly, including those of type $(7, 7; 6, 0)$ and minimum distance 8, type $(7, 7; 3, 1)$ and minimum distance 10, and $(7, 7; 3, 0)$ and minimum distance 12. Each of these codes provides a coloring of the hypercube H_{21}. Applying Theorem 3.1, we see that

$$\chi_7(21) \leqslant 2^{15},$$

$$\chi_9(21) \leqslant 2^{16},$$

and

$$\chi_{11}(21) \leqslant 2^{18}$$

whereas the upper bounds given by (3) are 2^{16}, 2^{18}, and 2^{20}, respectively. Additional examples may be found by applying the codes given in [1]. Perhaps the most interesting is the $\mathbb{Z}_2\mathbb{Z}_4$-linear code of type $(31, 31; 5, 1)$ and minimum distance 46 which allows us to conclude

$$\chi_{45}(93) \leqslant 2^{86},$$

as compared with the upper bound of 2^{91} given by (3). It is worth noting that while these codes are $\mathbb{Z}_2\mathbb{Z}_4$-linear, they are also binary linear codes. Hence, the bounds on the distance-ℓ chromatic number found in this example may also be obtained directly from (1).

4. Conclusion

In this paper, we consider colorings of the hypercube using nonlinear binary codes which are images of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. These give bounds on the distance-ℓ chromatic number of the n-dimensional hypercube for certain values of ℓ and n. Examples using particular codes, especially those from $\mathbb{Z}_4$-linear codes where nonlinear binary images which have better parameters than any binary linear code, are provided. While the results presented here are somewhat modest in scope, they lay the groundwork for further improvements as the study of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes matures.

References

[1] T. Abualrub, I. Siap, and N. Aydin, $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes, IEEE Trans. Inform. Theory 60 (2014), no. 3, 1508–1514.

[2] J. Borges, C. Fernández-Córdoba, J. Pujol, J. Rifà, M. Villanueva, $\mathbb{Z}_2\mathbb{Z}_4$-linear codes: generator matrices and duality, Des. Codes Cryptogr. 54 (2010), no. 2, 167–179.

[3] M. Bilal, J. Borges, S. T. Dougherty, and Fernández-Córdoba, Maximum distance separable codes over $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_4$, Des. Codes Cryptogr. 61 (2011), no. 1, 31–40.

[4] A. R. Calderbank and G. McGuire, Construction of a $(64, 2^{37}, 12)$ code via Galois rings, Des. Codes Cryptogr. 10 (1997), no. 2, 157–165.

[5] A. R. Calderbank, G. McGuire, P. V. Kumar, and T. Helleseth, Cyclic codes over $\mathbb{Z}_4$, locator polynomials, and Newton's identities, IEEE Trans. Inform. Theory 42 (1996), no. 1, 217-226.

[6] F.-W. Fu, S. Ling, and C.-P. Xing, New results on two hypercube coloring problems, Discrete Appl. Math. 161 (2013), no. 18, 2937–2945.

[7] M. Grassl, Bounds on the minimum distance of linear codes and quantum codes, Online available at http://www.codetables.de, Accessed on 2015-05-15.

[8] R. A. Hammons, P. V. Kumar, A. R. Calderbank, N. J. A. Sloane, and P. Solé, The Z_4-linearity of Kerdock, Preparata, Goethals, and related codes, IEEE Trans. Inform. Theory 40 (1994), no. 2, 301–319.

[9] M. Kiermaier and J. Zwanzger, New ring-linear codes from dualization in projective Hjelmslev geometries, Des. Codes Cryptogr. 66 (2013), no. 1-3, 39–55.

[10] M. Kiermaier and J. Zwanzger, A $\mathbb{Z}_4$-linear code of high minimum Lee distance derived from a hyperoval, Adv. Math. Commun. 5 (2011), no. 2, 275–286.

[11] D. S. Kim, D.-Z. Du, and P. M. Pardalos, A coloring problem on the n-cube, Discrete Appl. Math. 161 (2000), no. 103, 307–311.

[12] H. Q. Ngo, D.-Z. Du, and R. L. Graham, New bounds on a hypercube coloring problem, Inform. Process. Letters 84 (2002), 265–269.

[13] P. R. J. Östergård, On a hypercube coloring problem, J. Combin. Theory Ser. A 108 (2004), no. 2, 199–204.

[14] A. Pavan, P.-J. Wan, S.-R. Tong, D.-Z. Du, A new multihop lightwave network based on the generalized de-Brujin graph, Proc. on IEEE INFOCOM, 1996, 498-507.

[15] Z. Skupień, BCH codes and distance multi- or fractional colorings in hypercubes asymptotically, Discrete Math. 307 (2007), no. 7-8, 990–1000.

[16] Z. Skupień, Some maximum multigraphs and edge/vertex distance colourings, Discuss. Math. Graph Theory 15 (1995), no. 1, 89–106.

[17] P.-J. Wan, Near-optimal conflict free channel set assignments for an optical cluster-based hypercube network, J. Combin. Optimization 1 (1997), 179–186.

[18] G. M. Ziegler, Coloring Hamming graphs, optimal binary codes, and the 0/1-Borsuk problem in low dimensions, in: H. Alt (Ed.), Computational Discrete Mathematics, Springer, Berlin, 2001, pp. 159?171.