Numerical Approximation of a quasi-Newtonian Stokes Flow Problem with Defective Boundary Conditions

Vincent J. Ervin [*] Hyesuk Lee [*]

Abstract. In this article we study the numerical approximation of a quasi-Newtonian Stokes flow problem where only the flow rates are specified at the inflow and outflow boundaries. A variational formulation of the problem, using Lagrange multipliers to enforce the stated flow rates, is given. Existence and uniqueness of the solution to the continuous, and discrete, variational formulations is shown. An error analysis for the numerical approximation is also given. Numerical computations are included which demonstrate the approximation scheme studied.

Key words. quasi-Newtonian Stokes flow; Defective boundary condition; Power law fluid

AMS Mathematics subject classifications. 65N30

1 Introduction

In this paper we investigate the numerical approximation of a quasi-Newtonian Stokes flow problem with defective boundary conditions. Such problems arise in modeling viscoelastic fluid flow. Several examples are given in Section 2.1. For well-posedness of a Newtonian fluid flow problem suitable boundary conditions are required to uniquely define the solution. Perhaps the simplest of these is to specify the velocity at each point on the boundary of the domain. Often what is assumed is that the flow is *fully developed* at the inflow and outflow boundaries, which justifies a parabolic flow profile at these boundaries. Typically a *no slip* (i.e. velocity $= \mathbf{0}$) is assumed along the other portions of the boundary of the domain. However, in many physical problems the assumption of *fully developed* flow at the inflow and outflow is either unreasonable or highly questionable. Usually what is known in physical fluid flow problems are the various inflow and outflow flow rates.

In [6] Formaggia, Gerbeau, Nobile, and Quarteroni discuss the defective boundary condition problem for the time dependent Navier-Stokes equation. They introduce a Lagrange multiplier approach to enforce flow constraints at the inflow and outflow portions of the boundary. For the steady-state Stokes problem, they show existence and uniqueness of the solution for flow rates imposed using the Lagrange multiplier formulation. Herein we extend this work to analyze a quasi-Newtonian Stokes flow problem subject to specified inflow and outflow flow rates. We establish existence and uniqueness of solution for the continuous and discrete variational problems, and present an error analysis for the numerical approximation.

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Initially it is, perhaps, somewhat perplexing to note that for the uniqueness of solution to the variational problem for: (i) the Dirichlet problem we require $\acute{d}$ (the dimension of the space) conditions be specified at each point on the boundary, whereas (ii) the defective boundary condition problem only requires a single scalar be specified at inflow and outflow boundaries (and $\acute{d}$ conditions at other boundary points). This seeming anomaly is explained in Lemma 2.1 (see also [6] Proposition 2.1, [11] pg. 341). Specifically, the variational formulation for the defective boundary condition problem implicity imposes that across each of the inflow and outflow boundaries the total stress normal to the boundaries is a constant, and the extra stress lying in the surface of the inflow and outflow boundaries is zero.

In [11] Heywood, Rannacher, and Turek also investigated the defective boundary condition problem for the time dependent Navier-Stokes equations. They considered both the case of specified flow rates at the inflow and outflow boundaries, and also the case of the mean specified pressure at the inflow and outflow boundaries. For the specified flow rate problem, the formulation they considered (and proved existence of a steady state solution) involved the construction of suitable *flux-carrier* vector functions.

The numerical approximation of the quasi-Newtonian Stokes flow problem, with homogeneous boundary conditions has been previously studied in several papers [2, 5, 7, 13, 16].

This paper is organized as follows. In Sections 2.1 and 2.2 we describe the model problem, state our assumptions on the model, and introduce appropriate mathematical notation. We show in Section 2.3 that the corresponding variational formulation, in which the flow rate boundary conditions are weakly imposed using Lagrange multipliers, is well posed. A numerical approximation scheme is presented in Section 3, and its solution shown to exist. A priori error estimates for the numerical approximation are derived in Section 4. Numerical results are presented in Section 5.

2 Mathematical Model

Motivated by physical considerations we consider the numerical approximation of a three field, quasi-Newtonian Stokes flow problem with fixed flow-rate boundary conditions.

2.1 Problem Specification

Let Ω denote a bounded domain in $\mathbb{R}^{\acute{d}}$, $\acute{d} = 2$ or 3, whose boundary $\partial\Omega$ is decomposed into the union of Γ and several disjoint sections $S_1, S_2, \ldots, S_m$, $m \geq 2$.

We are interested in the numerical approximation of

$$\sigma = g(\mathbf{u}), \quad \text{in } \Omega, \tag{2.1}$$

$$-\nabla \cdot \sigma + \nabla p = \mathbf{f}, \quad \text{in } \Omega, \tag{2.2}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega, \tag{2.3}$$

$$\mathbf{u} = 0, \quad \text{on } \Gamma, \tag{2.4}$$

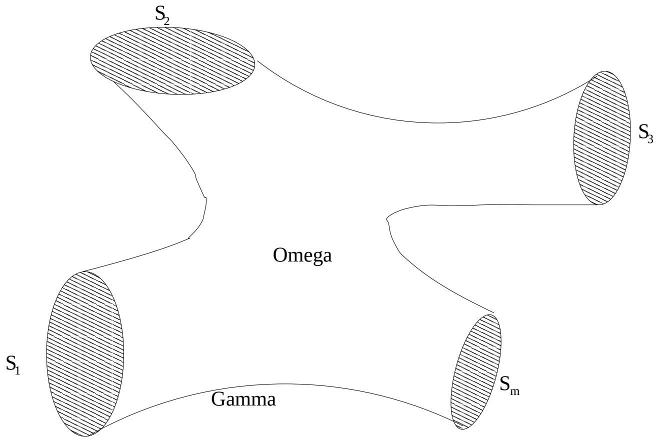

Figure 2.1: Illustration of flow domain.

subject to the specified flow rates across the surfaces S_i

$$\int_{S_i} \mathbf{u} \cdot \mathbf{n} \, ds \;=\; Q_i \ , \text{ for } i = 1, \ldots, m. \tag{2.5}$$

We use $\mathbf{n}$ to denote the outward (from Ω) normal to the surface.

Because of the incompressibility condition (2.3) it follows that

$$\sum_{i=1}^{m} Q_i \;=\; 0 \, . \tag{2.6}$$

Note that equations (2.1)-(2.5), can only determine the pressure, p, up to an arbitrary constant. Below we fix this constant by requiring p to have mean value 0 over Ω.

The general form of the (algebraic) constitutive equation we assume in our analysis (see **A1**, **A2**, **A3**, in Section 2) is motivated by the study of fluids having a *power law* constitutive equation, i.e.

$$\sigma \;=\; \nu_0 \, |d(\mathbf{u})|^{r-2} \, d(\mathbf{u}) \ , \quad \nu_0 > 0, \quad 1 < r < 2, \tag{2.7}$$

where σ denotes the extra stress tensor, $\mathbf{u}$ the fluid velocity, and $d(\mathbf{u}) := (\nabla \mathbf{u} + \nabla \mathbf{u}^T)/2$ the rate of deformation tensor.

The power law model has been used to model the viscosity of many polymeric solutions and melts over a considerable range of shear rates [10].

Other constitutive equations having a similar form to the power law model include [3, 13, 14]:
Ladyzhenskaya Law[12]:

$$\sigma \;=\; \nu_0 \,+\, \nu_1 \, |\nabla \mathbf{u}|^{r-2} \, d(\mathbf{u}) \ , \quad \nu_0 \ge 0, \ \nu_1 > 0, \ r > 1 \ , \tag{2.8}$$

used in modeling fluids with large stresses,
Carreau Law:

$$\sigma \;=\; \nu_0 \left(1 \,+\, |d(\mathbf{u})|^2\right)^{(r-2)/2} \, d(\mathbf{u}) \ , \quad \nu_0 > 0, \ r \ge 1 \ , \tag{2.9}$$

used in modeling visco-plastic flows and creeping flow of metals.

2.2 Notation/Assumptions

We made the following assumptions regarding the constitutive equation (2.1) for the stress σ.
A1: $g(\mathbf{u})$ is (formally) uniquely invertible to obtain

$$d(\mathbf{u}) = \breve{g}(\sigma)\sigma \quad , \quad (\text{ or } \nabla\mathbf{u} = \breve{g}(\sigma)\sigma)$$

and the inverse is continuous. For $G(\sigma) := \breve{g}(\sigma)\sigma$,

$$\mathbf{A2:}\ \ (G(s) - G(t)) : (s - t) \ \geq\ c|s - t|^{r'} \ , \qquad \forall s, t \in \mathbb{R}^{\acute{d}\times\acute{d}} \ , \tag{2.10}$$

$$\mathbf{A3:}\ \ |G(s) - G(t)| \ \leq\ M\left(|s| + |t|\right)^{r'-2}|s - t| \ , \qquad \forall s, t \in \mathbb{R}^{\acute{d}\times\acute{d}} \ . \tag{2.11}$$

For $r \in \mathbb{R}$, $r > 1$, we denote its unitary conjugate by r', satisfying $r^{-1} + r'^{-1} = 1$.

For problems of physical interest, $1 < r \leq 2$, e.g., shear thinning fluids. We therefore assume that $1 < r \leq 2$, and consequently, $2 \leq r' < \infty$.

Properties **A2** and **A3** imply that $G(\cdot)$ is strongly monotone, and Lipschitz continuous for bounded arguments [4].

We remark that differential constitutive models for viscoelastic fluids, such as the Oldroyd-B or Giesekus models, do not satisfy **A1**-**A3**.

Used in the analysis below are the following function spaces and norms.

$$T := \left(L^{r'}(\Omega)\right)^{\acute{d}\times\acute{d}}_{sym} \ = \ \left\{\tau = (\tau_{ij});\ \tau_{ij} = \tau_{ji};\ \tau_{ij} \in L^{r'}(\Omega);\ i,j = 1,\ldots,\acute{d}\right\},$$

with norm $\|\tau\|_T := \left(\int_\Omega |\tau|^{r'}\, d\Omega\right)^{1/r'}$.

$$X := \left\{\mathbf{v} \in \left(W_0^{1,r}(\Omega)\right)^{\acute{d}} \ : \ \mathbf{v}|_\Gamma = \mathbf{0}\right\},$$

with $W^{k,p}(\Omega)$ denoting the usual Sobolev space notation. We take for the norm on X, $\|v\|_X := \left(\int_\Omega |d(\mathbf{v})|^r\, d\Omega\right)^{1/r}$, which is equivalent to the usual $\|\cdot\|_{W^{1,r}}$ norm by the Poincaré-Friedrichs lemma.

$$P := L_0^{r'}(\Omega) = \left\{q \in L^{r'}(\Omega) : \int_\Omega q\, d\Omega = 0\right\},$$

with norm $\|q\|_P := \left(\int_\Omega |q|^{r'}\, d\Omega\right)^{1/r'}$.

We use V_X to denote the subspace of X defined by

$$V_X \ := \ \{\mathbf{v} \in X \ : \ \int_\Omega q\, \nabla\cdot\mathbf{v}\, d\Omega + \sum_{i=1}^{m} \beta_i \int_{S_i} \mathbf{v}\cdot\mathbf{n}\, ds \ = \ 0 \ , \forall (q,\beta) \in P \times \mathbb{R}^m\},$$

and let

$$V_T \ := \ \{\tau \in T \ : \int_\Omega \tau \ : \ d(\mathbf{v}) \ d\Omega \ = \ 0 \ , \forall \mathbf{v} \in V_X\} \, .$$

For a Banach space Y, Y' denotes its dual space with associated norm $\| \cdot \|_{Y'}$. For σ, τ tensors and $\mathbf{u}$, $\mathbf{v}$ vectors, we use $:$ and $\cdot$ to denote the scalar quantities $\sigma : \tau \ := \sum_{i=1}^{\acute{d}} \sum_{j=1}^{\acute{d}} \sigma_{ij}\tau_{ij}$ and $\mathbf{u} \cdot \mathbf{v} := \sum_{i=1}^{\acute{d}} \mathbf{u}_i \mathbf{v}_i$. We use $(\cdot, \cdot)$ to denote the L^2 inner product for functions (scalar, vector, or tensor) over Ω, and $\langle \cdot, \cdot \rangle$ to denote the duality pairing between a function space and its dual space.

2.3 Lagrange Multiplier Approach

We consider the following variational formulation to (2.1)-(2.5): *Given* $\mathbf{f} \in X'$, $Q \in \mathbb{R}^m$, *determine* $(\sigma, \mathbf{u}, p, \lambda) \in T \times X \times P \times \mathbb{R}^m$, *such that*

$$\begin{aligned} a(\sigma, \tau) \, - \, b(\tau, \mathbf{u}) \ &= \ 0 \, , \quad \forall \tau \in T \, , & (2.12) \\ b(\sigma, \mathbf{v}) \, - \, s(\mathbf{v}, (p, \lambda)) \ &= \ < \mathbf{f}, \mathbf{v} > , \quad \forall \mathbf{v} \in X \, , & (2.13) \\ s(\mathbf{u}, (q, \beta)) \ &= \ \sum_{i=1}^{m} Q_i \beta_i \, , \quad \forall (q, \beta) \in P \times \mathbb{R}^m \, , & (2.14) \end{aligned}$$

where

$$\begin{aligned} a(\sigma, \tau) \ &:= \ \int_\Omega \breve{g}(\sigma) \sigma : \tau \, d\Omega \ , & (2.15) \\ b(\tau, \mathbf{u}) \ &:= \ \int_\Omega \tau : d(\mathbf{u}) \, d\Omega \ , & (2.16) \\ s(\mathbf{v}, (p, \lambda)) \ &:= \ \int_\Omega p \, \nabla \cdot \mathbf{v} \, d\Omega \, + \, \sum_{i=1}^{m} \lambda_i \int_{S_i} \mathbf{v} \cdot \mathbf{n} \, ds. & (2.17) \end{aligned}$$

The *Lagrange multiplier* $\lambda \in \mathbb{R}^m$ is introduced to include the flow constraints (2.5) in the variational formulation, see [1],[6],[9].

Equivalence of the Differential Equations and Variational Formulations
The variational formulation is obtained by multiplying the differential equations by sufficiently smooth functions, integrating over the domain and, where appropriate, applying Green's theorem. The constraint equations are imposed weakly using Lagrange multipliers. For a smooth solution the steps used in deriving the variational equations can be reversed to show that equations (2.1)-(2.5) are satisfied. In addition we have that a smooth solution of (2.12)-(2.14) satisfies the following additional boundary conditions (see [6]).

For $\mathbf{n}$ the outward normal on S_i, express the extra stress vector on S_i, $\sigma \cdot \mathbf{n}$, as

$$\sigma \cdot \mathbf{n} \ = \ s_n \mathbf{n} \, + \, \mathbf{s}_T \, ,$$

where $s_n \ = \ (\sigma \cdot \mathbf{n}) \cdot \mathbf{n}$ and $\mathbf{s}_T \ = \ \sigma \cdot \mathbf{n} \, - \, s_n \mathbf{n}$. The scalar s_n represents the magnitude of the extra stress in the outward normal direction to S_i, and $\mathbf{s}_T$ the component of the extra stress vector which lies in the plane of S_i.

Lemma 2.1 *Any smooth solution of (2.12)-(2.14) satisfies the additional boundary conditions*

$$-p + s_n\,|_{S_i} \;=\; \lambda_i \quad \text{and} \quad \mathbf{s}_T\,|_{S_i} \;=\; \mathbf{0}\,, \quad i = 1, \ldots, m\,. \tag{2.18}$$

Proof: The proof follows as in [6].

■

Remark: The equations (2.1)-(2.5) do not uniquely define a solution, but rather a set of solutions. The variational formulation (2.12)–(2.14) chooses a solution from the solution set. Specifically, (2.12)–(2.14) chooses *the solution* which satisfies (2.18). A different variational formulation may result in a different selection for *the solution* from the solution set. (See, for example, [6].)

Unique Solvability of (2.12)–(2.14)

There are two main steps in showing that (2.12)–(2.14) is uniquely solvable. Step 1 involves showing that the (2.12)–(2.14) can be reduced to an equivalent problem involving only σ. Step 2 demonstrates that the stress is uniquely solvable. Used in Step 1 is the following lemma.

Lemma 2.2 *([8], Remark 4.2, pg. 61) Let* $(X, \|\cdot\|_X)$ *and* $(M, \|\cdot\|_M)$ *be two reflexive Banach spaces. Let* $(X', \|\cdot\|_{X'})$ *and* $(M', \|\cdot\|_{M'})$ *be their corresponding dual spaces. Let* $B : X \to M'$ *be a linear continuous operator and* $B' : M'' \to X'$ *the dual operator of* B*. Let* $V = ker(B)$ *be the kernel of* B*; we denote by* $V^o \subset X'$ *the polar set of* V : $V^o = \{x' \in X', < x', v >= 0\,, \forall v \in V\}$ *and* $\dot{B} : X/V \to M'$ *the quotient operator associated with* B*. The following three properties are equivalent:*
(i) $\exists c > 0$*, such that*

$$\inf_{q\in M} \sup_{v\in X} \frac{< Bv, q >}{\|q\|_M\,\|v\|_X} \geq c\,,$$

(ii) B' *is an isomorphism from* M'' *onto* V^o *and*

$$\|B'q\|_{X'} \geq C_B \|q\|_{M''} \;\forall q \in M''\,,$$

(iii) $\dot{B}$ *is an isomorphism from* X/V *onto* M' *and*

$$\|\dot{B}\dot{v}\|_{M'} \geq C_B \|\dot{v}\|_{X/V} \;\forall \dot{v} \in X/V\,.$$

As the first part of Step 1, we show that (p, λ) can be eliminated from (2.12)–(2.14). To do this we use the following *inf-sup* condition. (See also [17].)

Lemma 2.3 *There exists* $C_{PRX} > 0$ *such that*

$$\inf_{(q,\beta)\in P\times \mathbb{R}^m} \sup_{\mathbf{u}\in X} \frac{s(\mathbf{u}, (q, \beta))}{\|\mathbf{u}\|_X\,\|(q, \beta)\|_{P\times\mathbb{R}^m}} \geq C_{PRX}\,, \tag{2.19}$$

where $\|(q, \beta)\|_{P\times\mathbb{R}^m} \;:=\; \|q\|_P \,+\, \|\beta\|_{\mathbb{R}^m}$.

Proof: Fix $(q, \beta) \in P \times \mathbb{R}^m$ and let

$$\hat{q} = \frac{|q|^{r'/r-1}q}{\|q\|_P^{r'-1}} \quad , \quad \hat{\beta} = \frac{\beta}{\|\beta\|_{\mathbb{R}^m}} \quad . \tag{2.20}$$

Note that $(q\,,\hat{q}) \;=\; \|q\|_P$, $\|\hat{q}\|_{P'} = 1$, and $\hat{\beta}\cdot\beta \;=\; \|\beta\|_{\mathbb{R}^m}$ and $\|\hat{\beta}\|_{\mathbb{R}^m} = 1$.

Next, we introduce $\delta \in \mathbb{R}$ and $h \in W^{1-1/r\,,r}(\partial\Omega)$, a piecewise constant function, defined by

$$h \;=\; \begin{cases} \hat{\beta}_i/\mathtt{meas}(S_i) & \text{on} \quad S_i\,, i = 1,\ldots,m \\ 0 & \text{on} \quad \partial\Omega\backslash\cup_{i=1}^m S_i \end{cases}\,, \tag{2.21}$$

$$\delta \;=\; \left(\int_{\partial\Omega} h\,ds \;-\; \int_\Omega \hat{q}\,d\Omega\right)/\mathtt{meas}(\Omega)\,. \tag{2.22}$$

Consider the Neumann problem: *Given* $f \in W^{k,p}(\Omega), g \in W^{k+1-1/p,p}(\partial\Omega)\,,\; 1 < p < \infty$, *determine* $\dot{\phi} \in W^{k,p}(\Omega)/\mathbb{R}$ *(the quotient space) satisfying*

$$-\Delta\dot{\phi} \;=\; f\,, \quad \text{in}\ \Omega\,, \tag{2.23}$$

$$\frac{\partial\dot{\phi}}{\partial\mathbf{n}} \;=\; g\,, \quad \text{on}\ \partial\Omega\,. \tag{2.24}$$

From [8], pg. 15, with $\|\dot{\phi}\|_{W^{k+2,p}(\Omega)/\mathbb{R}} \;:=\; \inf_{\phi\in\dot{\phi}} \|\phi\|_{W^{k+2,p}(\Omega)}$,

we have the existence and uniqueness of $\dot{\phi}$ and a constant $C = C(k,p,\Omega)$ satisfying

$$\|\dot{\phi}\|_{W^{k+2,p}(\Omega)/\mathbb{R}} \;\le\; C\left(\|f\|_{W^{k,p}(\Omega)} \;+\; \|g\|_{W^{k+1-1/p,p}(\partial\Omega)}\right)\,. \tag{2.25}$$

Additionally, for $\phi \in \dot{\phi}$ we have that $\|\dot{\phi}\|_{W^{k+2,p}(\Omega)/\mathbb{R}} \equiv |\phi|_{W^{k+1,p}(\Omega)}$.

For the choices $f = \hat{q} \,+\, \delta$, $g = h$, $k = 0$, $p = r$, let $\dot{\phi}$ be given by (2.23)-(2.24).
Remark: The choice of the constant δ guarantees that the compatibility condition $\int_\Omega f\,d\Omega = \int_{\partial\Omega} g\,ds$ is satisfied.

We have that

$$\|\hat{q}\|_{W^{0,r}(\Omega)} \;=\; 1\ ,\ \text{(by construction)}, \tag{2.26}$$

$$\|h\|_{W^{1-1/r,r}(\partial\Omega)} \;\le\; C_1\,\|\hat{\beta}\|_{\mathbb{R}^m} \;=\; C_1\ , \tag{2.27}$$

as h is piecewise constant (and the equivalence of finite dimensional norms).

Also,

$$\int_\Omega \hat{q}\,d\Omega \;\le\; \|\hat{q}\|_{P'}\,\|1\|_P \;=\; C_2\ , \tag{2.28}$$

$$\int_{\partial\Omega} h\,ds \;\le\; \|\hat{\beta}\|_{\mathbb{R}^m}\,\|\mathbf{1}\|_{\mathbb{R}^m} \;=\; C_3\ , \tag{2.29}$$

and thus $\|\delta\|_{W^{0,r}(\Omega)} \;\le\; C_4$.

Letting $\mathbf{u} = -\nabla\phi$, $\phi \in \dot{\phi}$, we have that $\mathbf{u} \in \left(W^{1,r}(\Omega)\right)^d$ and from (2.25)–(2.29)

$$\|\mathbf{u}\|_{W^{1,r}(\Omega)} \;\le\; \|\dot{\phi}\|_{W^{2,r}(\Omega)} \;\le\; C\,(1 \,+\, C_4 \,+\, C_1) \;\le\; C_5\ . \tag{2.30}$$

Also, from (2.23)-(2.24), $\nabla \cdot \mathbf{u} = \hat{q} + \delta$, and $\int_{S_i} \mathbf{u} \cdot \mathbf{n} \, ds \; = \; \int_{S_i} h \, ds \; = \; \hat{\beta}_i$, $i = 1, \ldots, m$.

Hence,

$$\begin{aligned} s(\mathbf{u}, (q, \beta)) \; &= \; (\nabla \cdot \mathbf{u} \,, \, q) \; + \; \sum_{i=1}^{m} \beta_i \int_{S_i} \mathbf{u} \cdot \mathbf{n} \, ds \\ &= \; (\hat{q} + \delta, q) \; + \; \hat{\beta} \cdot \beta \\ &= \; \|q\|_P \; + \; \|\beta\|_{\mathbb{R}^m} \\ &= \; \|(q \,, \beta)\|_{P \times \mathbb{R}^m} \,, \end{aligned}$$

as $(\delta, q) = 0$ for $q \in P$ $(= L_0^{r'}(\Omega))$. Thus,

$$\sup_{\mathbf{u} \in W^{1,r}(\Omega)} \frac{s(\mathbf{u}, (q, \beta))}{\|(q \,, \beta)\|_{P \times \mathbb{R}^m} \|\mathbf{u}\|_{W^{1,r}(\Omega)}} \; \geq \; \frac{1}{C_5} \,,$$

from which (2.19) directly follows.

■

We now state and prove the existence and uniqueness of the solution to (2.12)–(2.14).

Theorem 2.1 *Given* $\mathbf{f} \in X'$ *and* $Q \in \mathbb{R}^m$, *there exists a unique* $(\sigma, \mathbf{u}, p, \lambda) \in T \times X \times P \times \mathbb{R}^m$ *satisfying (2.12)–(2.14).*

Proof: From Lemma 2.3 and Lemma 2.2(i),(iii), with the associations $X = X$, $M = P \times \mathbb{R}^m$, $B : X \rightarrow (P \times \mathbb{R}^m)'$ defined by

$$B(\mathbf{v}) \; := \; s(\mathbf{v}, (\cdot, \cdot)) \;,$$

$V = ker(B)$, we have that there exists $\dot{\mathbf{u}} \in X/V$, such that

$$s(\dot{\mathbf{u}}, (q, \beta)) \; = \; \sum_{i=1}^{m} Q_i \beta_i \;, \quad \forall (q, \beta) \in \; P \times \mathbb{R}^m \;,$$

with $\|\dot{\mathbf{u}}\|_{X/V} \leq 1/C_s \, \|Q\|_{\mathbb{R}^m}$.

Note: $\|\dot{\mathbf{u}}\|_{X/V} \; := \; \inf_{\mathbf{v} \in \dot{\mathbf{u}}} \|\mathbf{v}\|_X$.

As the cosets in X/V are closed, we can choose $\mathbf{u}_s \in \dot{\mathbf{u}}$ such that

$$\|\mathbf{u}_s\|_X = \|\dot{\mathbf{u}}\|_{X/V} \leq 1/C_s \, \|Q\|_{\mathbb{R}^m} \,. \tag{2.31}$$

Let $\mathbf{u} \; = \; \tilde{\mathbf{u}} \, + \, \mathbf{u}_s$. Then, solving (2.12)–(2.14) is equivalent to: *Find* $\sigma \in X, \tilde{\mathbf{u}} \in V_X$, *such that*

$$a(\sigma, \tau) \; - \; b(\tau, \tilde{\mathbf{u}}) \; = \; b(\tau, \mathbf{u}_s) \,, \quad \forall \tau \in T \,, \tag{2.32}$$

$$b(\sigma, \mathbf{v}) \; = \; < \mathbf{f}, \mathbf{v} > \,, \quad \forall \mathbf{v} \in V_X \,. \tag{2.33}$$

Now, note that for $\mathbf{v} \in X$, and $\tau = |d(\mathbf{v})|^{r-2} d(\mathbf{v}) \; \in \; T$, $\|\tau\|_T = \|\mathbf{v}\|_X^{r/r'}$, and

$$\frac{b(\tau, \mathbf{v})}{\|\tau\|_T} \; = \; \frac{\|\mathbf{v}\|_X^r}{\|\mathbf{v}\|_X^{r/r'}} \; = \; \|\mathbf{v}\|_X \,.$$

Thus

$$\inf_{\mathbf{v}\in X} \sup_{\tau\in T} \frac{b(\tau,\mathbf{v})}{\|\tau\|_T\|\mathbf{v}\|_X} \;\geq\; 1 \;, \tag{2.34}$$

i.e. $b(\tau,\mathbf{v})$ satisfies an inf-sup condition over $X \times T$.

As above, there exists $\sigma_b \in T$ such that

$$\begin{aligned} b(\sigma_b,\mathbf{v}) &= \;<\mathbf{f},\mathbf{v}>\;, \quad \forall \mathbf{v}\in X \;, \\ \text{with } \|\sigma_b\|_T &\leq \frac{1}{C_b}\|\mathbf{f}\|_{X'} \;. \end{aligned} \tag{2.35}$$

Let $\sigma \;=\; \tilde{\sigma} \,+\, \sigma_b$. Then, solving (2.32), (2.33) is equivalent to: *Find* $\tilde{\sigma} \in V_T$*, such that*

$$a(\tilde{\sigma}+\sigma_b,\tau) \;=\; b(\tau,\mathbf{u}_s) \quad \forall \tau\in V_T \,. \tag{2.36}$$

From assumptions **A2** and **A3** we have that $G(\tau) : V_T \rightarrow V_T'$ is a continuous, coercive, strictly monotone operator on a real, separable, reflexive Banach space [15]. Hence, there exists a unique $\tilde{\sigma} \in V_T$ satisfying (2.36). This then also uniquely determines $\sigma \in T$.

The inf-sup condition (2.34), together with (2.32), uniquely determine $\tilde{\mathbf{u}} \in V_X$ and hence also $\mathbf{u} = \tilde{\mathbf{u}} + \mathbf{u}_s \in X$.

Finally, the inf-sup condition (2.19) and equation (2.13) uniquely determine $p \in P$ and $\lambda \in \mathbb{R}^m$.

■

We now establish a bound for $\|\sigma\|_T$ which we use below in Section 4 in deriving a priori estimates for the numerical approximation. Estimates for $\mathbf{u}$, p, and λ can also be derived.

Corollary 2.1 *For $\sigma \in T$ satisfying (2.12)–(2.14) we have that there exists $C > 0$ such that*

$$\|\sigma\|_T \;\leq\; C\,(\|\mathbf{f}\|_{X'} \,+\, \|Q\|_{\mathbb{R}^m}^{r/r'}\,). \tag{2.37}$$

Proof: From (2.36), with the choice $\tau = \tilde{\sigma}$, we have

$$\begin{aligned} a(\tilde{\sigma}+\sigma_b,\tilde{\sigma}) &= b(\tilde{\sigma},\mathbf{u}_s) \\ &\leq 2^{-r'}\epsilon\,\|\tilde{\sigma}\|_T^{r'} \;+\; C\,\|\mathbf{u}_s\|_X^r \\ &\leq 2^{-r'}\epsilon\;(\|\tilde{\sigma}\,+\,\sigma_b\|_T \,+\, \|\sigma_b\|_T)^{r'} \;+\; C\,\|\mathbf{u}_s\|_X^r \\ &\leq \epsilon\|\tilde{\sigma}\,+\,\sigma_b\|_T^{r'} \,+\, \epsilon\|\sigma_b\|_T^{r'} \;+\; C\,\|\mathbf{u}_s\|_X^r \,. \end{aligned} \tag{2.38}$$

Using assumption (2.10),

$$\begin{aligned} a(\tilde{\sigma}+\sigma_b,\tilde{\sigma}) &= \int_\Omega \breve{g}(\tilde{\sigma}+\sigma_b)\,(\tilde{\sigma}+\sigma_b) : \tilde{\sigma}\,d\Omega \\ &= \int_\Omega \breve{g}(\tilde{\sigma}+\sigma_b)\,(\tilde{\sigma}+\sigma_b) : (\tilde{\sigma}+\sigma_b) \;-\; \int_\Omega \breve{g}(\tilde{\sigma}+\sigma_b)\,(\tilde{\sigma}+\sigma_b) : \sigma_b\,d\Omega \\ &\geq \int_\Omega c|\tilde{\sigma}+\sigma_b|^{r'}\,d\Omega \;-\; \|G(\tilde{\sigma}+\sigma_b)\|_{L^r}\,\|\sigma_b\|_T \\ &\geq c\|\tilde{\sigma}+\sigma_b\|_T^{r'} \;-\; \frac{c}{2M^r}\|G(\tilde{\sigma}+\sigma_b)\|_{L^r}^r \;-\; C\|\sigma_b\|_T^{r'} \,. \end{aligned} \tag{2.39}$$

We use (2.11) to estimate the second term on the RHS of (2.39).

$$
\begin{aligned}
\|G(\tilde{\sigma}+\sigma_b)\|_{L^r}^r &= \int_\Omega |G(\tilde{\sigma}+\sigma_b)|^r \, d\Omega \;\le\; M^r \int_\Omega \left((|\tilde{\sigma}+\sigma_b|+0)^{r'-2} \, |\tilde{\sigma}+\sigma_b-0| \right)^r d\Omega \\
&= M^r \int_\Omega |\tilde{\sigma}+\sigma_b|^{(r'-1)r} \, d\Omega \\
&= M^r \int_\Omega |\tilde{\sigma}+\sigma_b|^{r'} \, d\Omega \\
&= M^r \, \|\tilde{\sigma}+\sigma_b\|_T^{r'} \, . \qquad (2.40)
\end{aligned}
$$

Combining (2.38)-(2.40) we have that

$$
\left(\frac{c}{2} - \epsilon\right) \|\tilde{\sigma}+\sigma_b\|_T^{r'} \;\le\; C \left(\|\sigma_b\|_T^{r'} \;+\; \|\mathbf{u}_s\|_X^r \right) \, .
$$

As $\sigma = \tilde{\sigma} + \sigma_b$, and using the estimates (2.31) and (2.35), we obtain (2.37).

■

3 Discrete Approximation

We now describe the discrete approximation problem corresponding to (2.12)-(2.14) and show that the problem is well defined. Analogous to the continuous problem existence and uniqueness for the discrete problem relies on the approximating spaces satisfying suitable inf-sup conditions.

We begin by describing the finite element approximation framework used in the analysis.

Let $\Omega \subset \mathbb{R}^{\acute{d}}$ $(\acute{d} = 2, 3)$ be a polygonal domain and let $\mathcal{T}_h$ be a triangulation of Ω made of triangles (in $\mathbb{R}^2$) or tetrahedrals (in $\mathbb{R}^3$). Thus, the computational domain is defined by

$$
\Omega = \cup K; \quad K \in T_h.
$$

We assume that there exist constants c_1, c_2 such that

$$
c_1 h \le h_K \le c_2 \rho_K
$$

where h_K is the diameter of triangle (tetrahedral) K, ρ_K is the diameter of the greatest ball (sphere) included in K, and $h = \max_{K \in \mathcal{T}_h} h_K$. Let $P_k(A)$ denote the space of polynomials on A of degree no greater than k. Then we define the finite element spaces as follows.

$$
\begin{aligned}
T_h &:= \left\{ \tau \in T \cap C(\bar{\Omega})^{2\times 2} : \tau|_K \in P_l(K), \; \forall K \in \mathcal{T}_h \right\} , &(3.1)\\
X_h &:= \left\{ \mathbf{v} \in X \cap C(\bar{\Omega})^{2} : \mathbf{v}|_K \in P_k(K), \; \forall K \in \mathcal{T}_h \right\} , &(3.2)\\
P_h &:= \left\{ q \in P \cap C(\bar{\Omega}) : q|_K \in P_n(K), \; \forall K \in \mathcal{T}_h \right\} . &(3.3)
\end{aligned}
$$

We assume that the velocity-stress and the pressure-velocity spaces satisfy the following (typical) discrete inf-sup condition: *There exists constants C_{XTh}, $C_{PXh} > 0$, such that*

$$
\inf_{\mathbf{v} \in X_h} \sup_{\tau \in T_h} \frac{b(\tau, \mathbf{v})}{\|\tau\|_T \, \|\mathbf{v}\|_X} \;\ge\; C_{XTh} \, , \qquad (3.4)
$$

$$
\inf_{q \in P_h} \sup_{\mathbf{v} \in X_h} \frac{\int_\Omega q \, \nabla \cdot \mathbf{v} \, dA}{\|q\|_P \, \|\mathbf{v}\|_X} \;\ge\; C_{PXh} \, . \qquad (3.5)
$$

Discrete Approximation Problem: *Given* $\mathbf{f} \in X'$*, and* $Q \in \mathrm{I\!R}^m$*, determine* $(\sigma_h, \mathbf{u}_h, p_h, \lambda_h) \in T_h \times X_h \times P_h \times \mathrm{I\!R}^m$ *such that*

$$a(\sigma_h, \tau_h) - b(\tau_h, \mathbf{u}_h) = 0\,, \quad \forall \tau_h \in T_h\,, \tag{3.6}$$

$$b(\sigma_h, \mathbf{v}_h) - s(\mathbf{v}_h, (p_h, \lambda_h)) = < \mathbf{f}, \mathbf{v}_h >\,, \quad \forall \mathbf{v}_h \in X_h\,, \tag{3.7}$$

$$s(\mathbf{u}_h, (q_h, \beta_h)) = \sum_{i=1}^{m} Q_i \beta_i\,, \quad \forall (q_h, \beta_h) \in P_h \times \mathrm{I\!R}^m\,. \tag{3.8}$$

For the analysis a more general inf-sup condition than that given in (3.5) is needed. This is established using the following two lemmas. (See also [17].)

Lemma 3.1 *There exists* $C_{RXh} > 0$ *such that*

$$\inf_{\beta \in \mathrm{I\!R}^m} \sup_{\mathbf{v}_h \in X_h} \frac{\sum_{i=1}^m \beta_i \int_{S_i} \mathbf{v}_h \cdot \mathbf{n}\, ds}{\|\mathbf{v}_h\|_X \, \|\beta\|_{\mathrm{I\!R}^m}} \;\geq\; C_{RXh}\;. \tag{3.9}$$

Outline of Proof: From inspection of (3.9) we see that we would like to choose $\mathbf{v}_h \in X_h$ such that $\mathbf{v}_h \cdot \mathbf{n} = \beta_i$ on each S_i, and $\|\mathbf{v}_h\|_X \leq c\|\beta\|_{\mathrm{I\!R}^m}$. This is done by constructing a suitable $\mathbf{v}_{h,i}$, with $\mathbf{v}_{h,i}|_{S_j} = \mathbf{0},\ j \neq i$ and then letting $\mathbf{v}_h = \sum_i \mathbf{v}_{h,i}$.

We focus our attention on a single S_i. We will assume that on S_i $\mathbf{n}(x) \cdot \mathbf{n}(y) \geq c > 0$ at all points $x, y \in S_i$, for which $\mathbf{n}$ is defined. (That is, on S_i the normal $\mathbf{n}$ does not vary by more than 90 degrees. If the normal does vary by more than 90 degrees consider the surface as two surfaces.)

For ease of explanation, consider S_i as a straight line segment from $(0\,,0)$ to $(|S_i|, 0)$. Fix a *depth* d_i such that the rectangle $\mathcal{R}$ with vertices: $(|S_i|/6\,, 0)$, $(5|S_i|/6\,, 0)$, $(5|S_i|/6\,, d_i)$, $(|S_i|/6\,, d_i)$ lies in Ω. Introduce the labelling of the following points: $A := (|S_i|/6\,, 0)$, $B := (5|S_i|/6\,, 0)$, $C := (5|S_i|/6\,, d_i)$, $D := (|S_i|/6\,, d_i)$, $E := (|S_i|/3\,, 0)$, $F := (2|S_i|/3\,, 0)$, $G := (2|S_i|/3\,, d_i)$, and $H := (|S_i|/3\,, d_i)$.

Let $\tilde{\mathbf{n}} = \mathbf{n}|_{(|S_i|/2\,,0)}$ and g_i the continuous, piecewise bi-linear, function defined by $g_i|_{E,F} = \beta_i$, and $g_i|_{A,B,C,D,G,H} = 0$. (See Figure 3.1. In Figure 3.1 xi $= x/|S_i|$, and eta $= y/d$.).

We define the function $\tilde{\mathbf{v}}_i$ as $\tilde{\mathbf{v}}_i|_{\Omega \setminus \mathcal{R}} = \mathbf{0}$, and $\tilde{\mathbf{v}}_i|_{\mathcal{R}} = g_i \tilde{\mathbf{n}}$. Then,

$$\beta_i \int_{S_i} \tilde{\mathbf{v}}_i \cdot \mathbf{n}\, ds \;=\; \beta_i \int_E^F (\beta_i \tilde{\mathbf{n}}) \cdot \mathbf{n}\, ds \;\geq\; c_i\, \beta_i^2 |S_i|/3\;.$$

Also,

$$\begin{aligned} \|\tilde{\mathbf{v}}_i\|_X &= \left(\int_{\mathcal{R}} |\tilde{\mathbf{v}}_i|^r \, dA \;+ \int_{\mathcal{R}} |\nabla \tilde{\mathbf{v}}_i|^r \, dA \right)^{1/r} \\ &= |\beta_i| \left((r+2) d_i |S_i|/(3(r+1)^2) \;+\; 6^{r-2}\, 2\, d_i/(|S_i|^{r-1}(r+1)) \;+\; (6r+7)|S_i|/(18(r+1) d_i^{r-1}) \right)^{1/r}\;. \end{aligned}$$

Now, there exists h_0 such that for all $h \leq h_0$ there exists $\mathbf{v}_{h,i} \in \mathcal{T}_h$ such that $\|\tilde{\mathbf{v}}_i - \mathbf{v}_{h,i}\|_\infty \leq c_i\, \beta_i/6$ and $\|\tilde{\mathbf{v}}_i - \mathbf{v}_{h,i}\|_X \leq |\beta_i|$. Then,

$$\frac{\sum_{i=1}^m \beta_i \int_{S_i} \mathbf{v}_h \cdot \mathbf{n}\, ds}{\|\mathbf{v}_h\|_X} \;\geq\; \frac{\sum_{i=1}^m \beta_i \int_{S_i} \mathbf{v}_{h,i} \cdot \mathbf{n}\, ds}{\sum_{i=1}^m \|\mathbf{v}_{h,i}\|_X} \;\geq\; \frac{\sum_{i=1}^m \left(\beta_i \int_{S_i} \tilde{\mathbf{v}}_i \cdot \mathbf{n}\, ds - c_i \beta_i^2 |S_i|/6 \right)}{\sum_{i=1}^m \left(\|\tilde{\mathbf{v}}_i\|_X + \|\tilde{\mathbf{v}}_i - \mathbf{v}_{h,i}\|_X \right)}$$

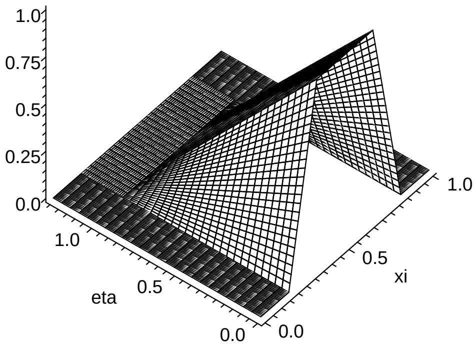

Figure 3.1: Plot of g_i/β_i.

$$\geq \frac{\sum_{i=1}^m c_i\,\beta_i^2 |S_i|/6}{\sum_{i=1}^m \hat{c}_i\,|\beta_i|} \;\geq\; C\|\beta\|\,,$$

from which (3.9) then follows.

■

Lemma 3.2 *For h sufficiently small, there exists $C_{PRXh} > 0$ such that*

$$\inf_{(q_h,\beta)\in P_h\times\mathbb{R}^m}\ \sup_{\mathbf{v}_h\in X_h} \frac{s(\mathbf{v}_h,(q_h,\beta))}{\|\mathbf{v}_h\|_X\,\|(q,\beta)\|_{P\times\mathbb{R}^m}} \geq C_{PRXh}\,. \tag{3.10}$$

Proof: Let $(p_h,\beta)\in P_h\times\mathbb{R}^m$. From Lemma 3.1, there exists $\hat{\mathbf{u}}_h\in X_h$ such that

$$\|\hat{\mathbf{u}}_h\|_X \;=\; \|\beta\|_{\mathbb{R}^m} \quad \text{and} \quad \frac{\sum_{i=1}^m \beta_i\int_{S_i}\hat{\mathbf{u}}_h\cdot\mathbf{n}\,ds}{\|\hat{\mathbf{u}}_h\|_X} \geq c_1\|\beta\|_{\mathbb{R}^m}\,. \tag{3.11}$$

Let $X_h^0 := \{\mathbf{v}_h \in X_h \;:\; \mathbf{v}_h|_{\partial\Omega} = \mathbf{0}\}$, and consider the (discrete) power-law problem: *Determine* $\tilde{\mathbf{u}}_h\in X_h^0$, $\tilde{p}_h\in P_h$ *such that*

$$(|d(\tilde{\mathbf{u}}_h)|^{r-2}d(\tilde{\mathbf{u}}_h)\,,\,d(\mathbf{v})) \;-\; (\tilde{p}_h\,,\,\nabla\cdot\mathbf{v}) \;=\; 0\,,\quad \forall\mathbf{v}\in X_h^0\,, \tag{3.12}$$

$$(q\,,\,\nabla\cdot\tilde{\mathbf{u}}_h) \;=\; (q\,,\,\|p_h\|_P^{1-r'/r}\,|p_h|^{r'/r-1}\,p_h \;-\; \nabla\cdot\hat{\mathbf{u}}_h)\,,\quad \forall q\in P_h\,. \tag{3.13}$$

Note that $\|p_h\|_P^{1-r'/r}\,|p_h|^{r'/r-1}\,p_h \,-\, \nabla\cdot\hat{\mathbf{u}}_h \;\in\; L^r(\Omega)$.

Existence and uniqueness of $\tilde{\mathbf{u}}_h\in X_h^0$, $\tilde{p}_h\in P_h$ satisfying (3.12),(3.13) follows analogous to the proof of Theorem 2.1. (See also [8, 2]).

From (3.12),(3.13) with the choices $\mathbf{v}=\tilde{\mathbf{u}}_h$, and $q=\tilde{p}_h$,

$$\|\tilde{\mathbf{u}}_h\|_X^r \;=\; (|d(\tilde{\mathbf{u}}_h)|^{r-2}d(\tilde{\mathbf{u}}_h)\,,\,d(\tilde{\mathbf{u}}_h)) \;=\; (\tilde{p}_h\,,\,\nabla\cdot\tilde{\mathbf{u}}_h)$$

$$
\begin{aligned}
&= (\tilde{p}_h\,,\, \|p_h\|_P^{1-r'/r}\, |p_h|_P^{r'/r-1}\, p_h \,-\, \nabla\cdot\hat{\mathbf{u}}_h) \\
&\le \|\tilde{p}_h\|_P \left(\|p_h\|_P^{1-r'/r}\, \| |p_h|_P^{r'/r-1}\, p_h\|_{L^r} \,+\, \|\nabla\cdot\hat{\mathbf{u}}_h\|_{L^r} \right) \\
&\le \|\tilde{p}_h\|_P \,(\|p_h\|_P \,+\, C\,\|\hat{\mathbf{u}}_h\|_X) \\
&= \|\tilde{p}_h\|_P \left(\|p_h\|_P \,+\, \|\beta\|_{\mathbb{R}^m}\right)\,.
\end{aligned} \tag{3.14}
$$

Also, from the inf-sup condition for spaces X_h^0 and P_h we have

$$
\begin{aligned}
c\,\|\tilde{p}_h\|_P &\le \sup_{\mathbf{v}\in X_h^0} \frac{(\tilde{p}_h\,,\,\nabla\cdot\mathbf{v})}{\|\mathbf{v}\|_X} \\
&= \sup_{\mathbf{v}\in X_h^0} \frac{(|d(\tilde{\mathbf{u}}_h)|^{r-2} d(\tilde{\mathbf{u}}_h)\,,\, d(\mathbf{v}))}{\|\mathbf{v}\|_X} \\
&\le \sup_{\mathbf{v}\in X_h^0} \frac{(\| |d(\tilde{\mathbf{u}}_h)|^{r-2} d(\tilde{\mathbf{u}}_h)\|_{L^{r'}}\, \|d(\mathbf{v})\|_{L^r}}{\|\mathbf{v}\|_X} \\
&= \| |d(\tilde{\mathbf{u}}_h)|^{r-2} d(\tilde{\mathbf{u}}_h)\|_{L^{r'}} \\
&= \|\tilde{\mathbf{u}}_h\|_X^{r/r'}\,.
\end{aligned} \tag{3.15}
$$

Combining (3.14) and (3.15) we have the estimate

$$
\|\tilde{\mathbf{u}}_h\|_X \;\le\; \left(\|p_h\|_P \,+\, C\,\|\beta\|_{\mathbb{R}^m}\right)\,. \tag{3.16}
$$

Let $\mathbf{u}_h \;=\; \tilde{\mathbf{u}}_h \,+\, \hat{\mathbf{u}}_h$. Then, using (3.13) and (3.11)

$$
\begin{aligned}
s(\mathbf{u}_h, (p_h,\beta)) &= \int_\Omega p_h\,\nabla\cdot\tilde{\mathbf{u}}_h\,d\Omega \,+\, \int_\Omega p_h\,\nabla\cdot\hat{\mathbf{u}}_h\,d\Omega \,+\, \sum_{i=1}^m \beta_i \int_{S_i} \tilde{\mathbf{u}}_h\cdot\mathbf{n}\,ds \\
&\quad +\, \sum_{i=1}^m \beta_i \int_{S_i} \hat{\mathbf{u}}_h\cdot\mathbf{n}\,ds \\
&= \int_\Omega p_h\,\|p_h\|_P^{1-r'/r}\,|p_h|^{r'/r-1}\,p_h\,d\Omega \,+\, \sum_{i=1}^m \beta_i \int_{S_i} \hat{\mathbf{u}}_h\cdot\mathbf{n}\,ds \\
&\ge c\left(\|p_h\|_P^2 \,+\, \|\beta\|_{\mathbb{R}^m}^2\right)\,.
\end{aligned} \tag{3.17}
$$

Thus, using (3.11),(3.16), we have

$$
\begin{aligned}
\sup_{\mathbf{v}_h\in X_h} \frac{s(\mathbf{v}_h, (p_h,\beta))}{\|\mathbf{v}_h\|_X} &\ge \frac{s(\mathbf{u}_h, (p_h,\beta))}{\|\mathbf{u}_h\|_X} \\
&\ge C\,\left(\|p_h\|_P \,+\, \|\beta\|_{\mathbb{R}^m}\right)\,,
\end{aligned}
$$

from which (3.10) immediately follows.

■

We now state and prove the existence and uniqueness of solutions to (3.6)–(3.8).

Theorem 3.1 *Given* $\mathbf{f} \in X'$ *and* $Q \in \mathbb{R}^m$*, there exists a unique* $(\sigma_h, \mathbf{u}_h, p_h, \lambda_h) \in T_h \times X_h \times P_h \times \mathbb{R}^m$ *satisfying (3.6)–(3.8). In addition,*

$$\|\sigma_h\|_T \ \leq \ C\,(\|\mathbf{f}\|_{X'} \,+\, \|Q\|_{\mathbb{R}^m}^{r/r'}\,). \tag{3.18}$$

Proof: With the inf-sup conditions given in (3.4) and (3.10) the proof of existence follows exactly as for the continuous problem in Theorem 2.1. Similarly, the norm estimate for σ_h follows as that for σ given in Corollary 2.1.

■

4 A Priori Error Estimate

In this section we derive an error estimate for the error in the approximation $(\sigma_h\,,\,\mathbf{u}_h\,,\,p_h\,,\,\lambda_h)$ satisfying (3.6)–(3.8), and $(\sigma\,,\,\mathbf{u}\,,\,p\,,\,\lambda)$ satisfying (2.12)–(2.14).

The proof of the estimates gives in Theorem 4.1 follow along the same lines as the proofs for the existence and uniqueness, except for the error estimates we work *backwards.* The procedure to establish existence and uniqueness was to reduce the problem to an equivalent problem for σ (or σ_h) on a subspace of the solution space. To obtain the error estimates we begin by considering the determining equations for σ_h, u_h, over a subspace. Using the coercivity and continuity assumptions (2.10), (2.11), an error estimate for $\|\sigma - \sigma_h\|$ over the subspace is constructed. We then show that the estimate over the subspace can be extended to the entire solution space.

Useful in the analysis below is the following inf-sup condition which follows from (3.4) and (3.10).

Lemma 4.1 *For* h *sufficiently small, there exists a constant* $C_{XTPRh} > 0$ *such that*

$$\inf_{\mathbf{v}\in X_h} \ \sup_{(\tau,q,\beta)\in T_h\times P_h\times\mathbb{R}^m} \frac{b(\tau,\mathbf{v}) \,-\, s(\mathbf{v},(q,\beta))}{\|(\tau,q,\beta)\|_{T\times P\times\mathbb{R}^m}\,\|\mathbf{v}\|_X} \ \geq \ C_{XTPRh}\,, \tag{4.1}$$

where $\|(\tau,q,\beta)\|_{T\times P\times\mathbb{R}^m} \ := \ \|\tau\|_T \,+\, \|q\|_P \,+\, \|\beta\|_{\mathbb{R}^m}$*.*

Proof: For $\mathbf{v} \in X_h$, from (3.4) there exists τ_v such that

$$b(\tau_v,\mathbf{v}) \geq \frac{C_{XTh}}{2}\,\|\tau_v\|_T\,\|\mathbf{v}\|_X\,. \tag{4.2}$$

We now consider two cases. Firstly, if $s(\mathbf{v},(q,\beta)) = 0\,,\ \forall (q,\beta) \in P_h \times \mathbb{R}^m$, then (4.1) follows immediately from (4.2). Otherwise, from the definition of $s(\mathbf{v},(q,\beta))$, there exists $(q_v,\beta_v) \in P_h\times\mathbb{R}^m$ such that $s(\mathbf{v},(q_v,\beta_v)) < 0$ and $\|(q_v,\beta_v)\|_{P_h\times\mathbb{R}^m} \ = \ \|\tau_v\|_T$. Thus,

$$\begin{aligned}\sup_{(\tau,q,\beta)\in T_h\times P_h\times\mathbb{R}^m} \frac{b(\tau,\mathbf{v}) \,-\, s(\mathbf{v},(q,\beta))}{\|(\tau,q,\beta)\|_{T\times P\times\mathbb{R}^m}} \ &\geq \ \frac{b(\tau_v,\mathbf{v}) \,-\, s(\mathbf{v},(q_v,\beta_v))}{\|(\tau_v,q_v,\beta_v)\|_{T\times P\times\mathbb{R}^m}} \\ &\geq \ \frac{C_{XTh}\,\|\tau_v\|_T\,\|\mathbf{v}\|_X}{2\,(\|\tau_v\|_T \,+\, \|\tau_v\|_T)}\,,\end{aligned}$$

from which (4.2) then follows.

■

Theorem 4.1 *For $(\sigma\,,\,\mathbf{u}\,,\,p\,,\,\lambda)$ satisfying (2.12)–(2.14) and $(\sigma_h\,,\,\mathbf{u}_h\,,\,p_h\,,\,\lambda_h)$ satisfying (3.6)–(3.8), for h sufficiently small, we have that there exists a constant $C > 0$ such that*

$$\|\sigma - \sigma_h\|_T^{r'} \;\leq\; C \left(\inf_{\tau_h \in T_h} \left(\|\sigma - \tau_h\|_T^r \,+\, \|\sigma - \tau_h\|_T^{r'} \right) \,+\, \inf_{\mathbf{v}_h \in X_h} \|\mathbf{u} - \mathbf{v}_h\|_X^r \,+\, \inf_{q_h \in P_h} \|p - q_h\|_P^{r'} \right) \tag{4.3}$$

$$\|\mathbf{u} - \mathbf{u}_h\|_X \;\leq\; C \left(\|\sigma - \sigma_h\|_T \,+\, \inf_{\mathbf{v}_h \in X_h} \|\mathbf{u} - \mathbf{v}_h\|_X \right) , \tag{4.4}$$

$$\|p - p_h\|_P \;+\; \|\lambda - \lambda_h\|_{\mathbb{R}^m} \;\leq\; C \left(\|\sigma - \sigma_h\|_T \,+\, \inf_{q_h \in P_h} \|p - q_h\|_P \right) . \tag{4.5}$$

Proof: We have that $(\sigma_h\,,\,\mathbf{u}_h\,,\,p_h\,,\,\lambda_h)$ satisfies

$$a(\sigma_h, \tau_h) \,-\, b(\tau_h, \mathbf{u}_h) \;=\; 0\,, \quad \forall \tau_h \in T_h\,, \tag{4.6}$$

$$b(\sigma_h, \mathbf{v}_h) \,-\, s(\mathbf{v}_h, (p_h, \lambda_h)) \;=\; <\mathbf{f}, \mathbf{v}_h>\,, \quad \forall \mathbf{v}_h \in X_h\,, \tag{4.7}$$

$$s(\mathbf{u}_h, (q_h, \beta_h)) \;=\; \sum_{i=1}^m Q_i\,\beta_i\,, \quad \forall (q_h, \beta) \in P_h \times \mathbb{R}^m\,. \tag{4.8}$$

Introduce the affine subspaces $\tilde{X}_h \subset X_h$, $\tilde{K}_h$ defined by

$$\tilde{X}_h \;:=\; \{\mathbf{v}_h \in X_h \,:\, s(\mathbf{v}_h, (q_h, \beta)) \;=\; \sum_{i=1}^m Q_i\,\beta_i\,, \;\; \forall (q_h, \beta_h) \in P_h \times \mathbb{R}^m\}\,, \tag{4.9}$$

$$\tilde{K}_h \;:=\; \{\tau_h \in T_h \,:\, b(\tau_h, \mathbf{v}_h) \,-\, s(\mathbf{v}_h, (p_h, \lambda_h)) \;=\; <\mathbf{f}, \mathbf{v}_h>\,, \;\; \forall \mathbf{v}_h \in \tilde{X}_h\}\,. \tag{4.10}$$

Note that $\sigma_h \in \tilde{K}_h$ and $\mathbf{u}_h \in \tilde{X}_h$.

From (2.10),(2.11) we have

$$\begin{aligned} c\|\sigma - \sigma_h\|_T^{r'} \;&\leq\; a(\sigma, \sigma - \sigma_h) - a(\sigma_h, \sigma - \sigma_h) \\ &=\; a(\sigma, \sigma - \tau_h) - a(\sigma_h, \sigma - \tau_h) \;+\; a(\sigma, \tau_h - \sigma_h) - a(\sigma_h, \tau_h - \sigma_h) \\ &\leq\; \int_\Omega M(|\sigma| + |\sigma_h|)^{r'-2} |\sigma - \sigma_h| |\sigma - \tau_h| \, d\Omega \;+\; a(\sigma, \tau_h - \sigma_h) - a(\sigma_h, \tau_h - \sigma_h) \end{aligned} \tag{4.11}$$

Now, noting that $1 < r \leq 2$, and hence $r'/r \geq 1$,

$$\begin{aligned} \int_\Omega M(|\sigma| + |\sigma_h|)^{r'-2} |\sigma - \sigma_h| |\sigma - \tau_h| \, d\Omega \;&\leq\; \left(\int_\Omega M^r (|\sigma| + |\sigma_h|)^{(r'-2)r} \, |\sigma - \tau_h|^r \, d\Omega \right)^{1/r} \; \|\sigma - \sigma_h\|_T \\ &\leq\; \epsilon \, \|\sigma - \sigma_h\|_T^{r'} \;+\; C M^r \int_\Omega 2^{(r'-2)r} \left(|\sigma|^{(r'-2)r} + |\sigma_h|^{(r'-2)r} \right) \, |\sigma - \tau_h|^r \, d\Omega \\ &\leq\; \epsilon \, \|\sigma - \sigma_h\|_T^{r'} \\ &\quad + C \left(\int_\Omega \left(|\sigma|^{(r'-2)r} + |\sigma_h|^{(r'-2)r} \right)^{r'/(r'-r)} d\Omega \right)^{(r'-r)/r'} \left(\int_\Omega |\sigma - \tau_h|^{r'} \, d\Omega \right)^{r/r'} \\ &\leq\; \epsilon \, \|\sigma - \sigma_h\|_T^{r'} \;+\; C \left(\|\sigma\|_T^{r'} + \|\sigma_h\|_T^{r'} \right)^{(r'-r)/r'} \; \|\sigma - \tau_h\|_T^r \\ &\leq\; \epsilon \, \|\sigma - \sigma_h\|_T^{r'} \;+\; C \, \|\sigma - \tau_h\|_T^r \,. \end{aligned} \tag{4.12}$$

With the choice $\tau_h \in \tilde{K}_h$, using (2.12) and (4.9)

$$\begin{aligned}
a(\sigma, \tau_h - \sigma_h) \,-\, a(\sigma_h, \tau_h - \sigma_h) &= b(\tau_h - \sigma_h, \mathbf{u}) \,-\, b(\tau_h - \sigma_h, \mathbf{u}_h) \\
&= b(\tau_h - \sigma_h, \mathbf{u}) \quad (\text{since } \tau_h \text{ and } \sigma_h \text{ are in } \tilde{K}_h) \\
&= b(\tau_h - \sigma_h, \mathbf{u} - \mathbf{v}_h)\,, \quad (\text{for } \mathbf{v}_h \in \tilde{X}_h) \\
&= \int_\Omega (\tau_h - \sigma_h) : d(\mathbf{u} - \mathbf{v}_h)\, d\Omega \\
&= \int_\Omega (\sigma - \sigma_h) : d(\mathbf{u} - \mathbf{v}_h)\, d\Omega \,+\, \int_\Omega (\tau_h - \sigma) : d(\mathbf{u} - \mathbf{v}_h)\, d\Omega \\
&\leq \|\sigma - \sigma_h\|_T \,\|\mathbf{u} - \mathbf{v}_h\|_X \,+\, \|\sigma - \tau_h\|_T \,\|\mathbf{u} - \mathbf{v}_h\|_X \\
&\leq \epsilon\, \|\sigma - \sigma_h\|_T^{r'} \,+\, C\left(\|\sigma - \tau_h\|_T^{r'} \,+\, \|\mathbf{u} - \mathbf{v}_h\|_X^{r}\right). \qquad (4.13)
\end{aligned}$$

Combining (4.14)–(4.16) gives an error bound for $\|\sigma - \sigma_h\|_T$ in terms of the *best approximations* of σ and $\mathbf{u}$ in the sets $\tilde{K}_h$ and $\tilde{X}_h$, respectively. Next we show that we can *lift* these *best approximations* from $\tilde{K}_h$ and $\tilde{X}_h$ to $T_h \times X_h$. This is done in two steps. Firstly *lifting* from $\tilde{K}_h$ to $\tilde{W}_h$, and then using the discrete inf-sup condition to go from $\tilde{W}_h$ to $T_h \times X_h$.

Let

$$\tilde{W}_h := \{(\tau_h, q_h) \in T_h \times P_h \,:\, b(\tau_h, \mathbf{v}_h) \,-\, s(\mathbf{v}_h, (q_h, \lambda_h)) \;=\, <\mathbf{f}, \mathbf{v}_h>, \quad \forall \mathbf{v}_h \in X_h\}\,. \qquad (4.14)$$

Note that if (τ_h, q_h) is in $\tilde{W}_h$ then τ_h is in $\tilde{K}_h$. Hence,

$$\inf_{\tau_h \in K_h} \|\sigma - \tau_h\|_T \;\leq\; \inf_{(\tau_h, q_h) \in \tilde{W}_h} \|(\sigma\,,\, p) \,-\, (\tau_h\,,\, q_h)\|_{T \times P}\,. \qquad (4.15)$$

From the inf-sup conditions (4.1) we have that there exist operators $\Pi_1 \,:\, T \to T_h$, and $\Pi_2 \,:\, P \to P_h$ such that

$$b(\tau \,-\, \Pi_1 \tau, \mathbf{v}_h) \,-\, s(\mathbf{v}_h, (q - \Pi_2 q, \lambda_h)) \;=\; 0\,, \quad \forall \mathbf{v}_h \in X_h\,, \qquad (4.16)$$

and

$$\|(\Pi_1 \tau\,, \Pi_2 q)\|_{T \times P} \;\leq\; \tilde{C} \|(\tau\,, q)\|_{T \times P}\,, \quad \forall (\tau\,,\, q) \in T \times P\,. \qquad (4.17)$$

Consider $(\tau_h\,,\, q_h) \in T_h \times P_h$ and introduce $\tilde{\sigma} := \tau_h - \Pi_1(\tau_h - \sigma)$, and $\tilde{p} := q_h - \Pi_2(q_h - p)$. Then, for all $\mathbf{v}_h \in X_h$

$$\begin{aligned}
b(\tilde{\sigma}, \mathbf{v}_h) \,-\, s(\mathbf{v}_h, (\tilde{p}, \lambda_h)) &= b(\sigma, \mathbf{v}_h) \,-\, s(\mathbf{v}_h, (p, \lambda_h)) \\
&= \;<\mathbf{f}, \mathbf{v}_h>\,,
\end{aligned}$$

which implies $(\tilde{\sigma}, \tilde{p}) \in \tilde{W}_h$.

Also, using (4.20),

$$\begin{aligned}
\|(\tilde{\sigma}, \tilde{p}) \,-\, (\tau_h, q_h)\|_{T \times P} &= \|(\Pi_1(\sigma - \tau_h)\,,\, \Pi_2(p - q_h))\|_{T \times P} \\
&\leq \tilde{C} \|(\sigma - \tau_h\,,\, p - q_h)\|_{T \times P}\,. \qquad (4.18)
\end{aligned}$$

With $(\tilde{\sigma}, \tilde{p})$ as defined above, using (4.20),(4.21) and the triangle inequality,

$$\begin{aligned}
\inf_{(\tau_h,q_h)\in\tilde{W}_h} \|(\sigma\,,\,p)\;-\;(\tau_h\,,\,q_h)\|_{T\times P} \;&\leq\; \inf_{(\tau_h,q_h)\in T_h\times P_h} \|(\sigma\,,\,p)\;-\;(\tilde{\sigma},\tilde{p})\|_{T\times P} \\
&\leq\; \inf_{(\tau_h,q_h)\in T_h\times P_h} \left(\|(\sigma\,,\,p)\;-\;(\tau_h\,,\,q_h)\|_{T\times P}\;+\;\|(\tilde{\sigma},\tilde{p})\;-\;(\tau_h\,,\,q_h)\|_{T\times P}\right) \\
&\leq\; (1\,+\,\tilde{C}) \inf_{(\tau_h,q_h)\in T_h\times P_h} \|(\sigma\,,\,p)\;-\;(\tau_h\,,\,q_h)\|_{T\times P}\,. \qquad (4.19)
\end{aligned}$$

Using an analogous argument with the inf-sup condition (3.10) it is straight forward to show that

$$\inf_{\mathbf{v}_h\in\tilde{X}_h} \|\mathbf{u}\;-\;\mathbf{v}_h\|_X \;\leq\; C \inf_{\mathbf{v}_h\in X_h} \|\mathbf{u}\;-\;\mathbf{v}_h\|_X\,. \qquad (4.20)$$

Combining (4.14)–(4.16), (4.18), (4.22), and (4.23) we then have

$$\|\sigma-\sigma_h\|_T^{r'} \;\leq\; C\left(\inf_{\tau_h\in T_h}\left(\|\sigma-\tau_h\|_T^{r}\,+\,\|\sigma-\tau_h\|_T^{r'}\right)\;+\;\inf_{\mathbf{v}_h\in X_h}\|\mathbf{u}\;-\;\mathbf{v}_h\|_X^{r}\;+\;\inf_{q_h\in P_h}\|p\;-\;q_h\|_P^{r'}\right)\,.$$

To obtain the error estimate for the velocity we use (3.4). We have that

$$\begin{aligned}
C_{XTh}\,\|\mathbf{u}_h-\mathbf{v}_h\|_X \;&\leq\; \sup_{\tau_h\in T_h} \frac{b(\tau_h\,,\,\mathbf{u}_h-\mathbf{v}_h)}{\|\tau_h\|_T} \\
&=\; \sup_{\tau_h\in T_h} \frac{b(\tau_h\,,\,\mathbf{u}_h-\mathbf{u})\;+\;b(\tau_h\,,\,\mathbf{u}-\mathbf{v}_h)}{\|\tau_h\|_T} \\
&\leq\; \sup_{\tau_h\in T_h} \frac{a(\sigma_h\,,\,\tau_h)\;-\;a(\sigma\,,\,\tau_h)}{\|\tau_h\|_T}\;+\;\|\mathbf{u}-\mathbf{v}_h\|_X\,. \qquad (4.21)
\end{aligned}$$

Proceeding as in the estimate (4.15), we have that

$$\begin{aligned}
a(\sigma_h\,,\,\tau_h)\;-\;a(\sigma\,,\,\tau_h) \;&=\; \int_\Omega \left(\breve{g}(\sigma_h)\sigma_h-\breve{g}(\sigma)\sigma\right)\,:\,\tau_h\,d\Omega \\
&\leq\; \int_\Omega M\left(|\sigma_h|\,+\,|\sigma|\right)^{r'-2}|\sigma-\sigma_h|\,:\,\tau_h\,d\Omega \\
&\leq\; C\,\|\sigma-\sigma_h\|_T\,\|\tau_h\|_T\,. \qquad (4.22)
\end{aligned}$$

Combining (4.24) and (4.25) yields

$$\|\mathbf{u}_h-\mathbf{v}_h\|_X \;\leq\; C\,\left(\|\sigma\;-\;\sigma_h\|_T\;+\;\|\mathbf{u}\;-\;\mathbf{v}_h\|_X\right)\,.$$

An application of the triangle inequality then establishes (4.5).

The error estimate for the pressure and the "Lagrange multipliers" is obtained using the inf-sup condition (3.10), the trace theorem, and the equivalence of norms in $\mathbb{R}^m$. We have that

$$\begin{aligned}
C_{PRXh}\left(\|p_h-q_h\|_P\,+\,\|\lambda_h-\beta_h\|_{\mathbb{R}^m}\right) \;&\leq\; \sup_{\mathbf{v}_h\in X_h} \frac{s(\mathbf{v}_h\,,\,(p_h-q_h\,,\,\lambda_h-\beta_h))}{\|\mathbf{v}_h\|_X} \\
&=\; \sup_{\mathbf{v}_h\in X_h} \frac{s(\mathbf{v}_h\,,\,(p_h-p\,,\,\lambda_h-\lambda))\;+\;s(\mathbf{v}_h\,,\,(p-q_h\,,\,\lambda-\beta_h))}{\|\mathbf{v}_h\|_X}
\end{aligned}$$

$$
\begin{aligned}
&\le \sup_{\mathbf{v}_h \in X_h} \frac{b(\sigma\,,\,\mathbf{v}_h) \;-\; b(\sigma_h\,,\,\mathbf{v}_h)}{\|\mathbf{v}_h\|_X} \\
&\quad + \sup_{\mathbf{v}_h \in X_h} \frac{\int_\Omega (p - q_h)\,\nabla\cdot\mathbf{v}_h\,d\Omega \;+\; \sum_{i=1}^m (\lambda_i - \beta_{h,i}) \int_{S_i} \mathbf{v}_h\cdot\mathbf{n}\,ds}{\|\mathbf{v}_h\|_X} \\
&\le \sup_{\mathbf{v}_h \in X_h} \frac{\int_\Omega (\sigma - \sigma_h) \,:\, d(\mathbf{v}_h)\,d\Omega}{\|\mathbf{v}_h\|_X} \;+\; C\left(\|p - q_h\|_P \;+\; \|\lambda - \beta_h\|_{\mathrm{I\!R}^m}\right) \\
&\le \|\sigma - \sigma_h\|_T \;+\; C\left(\|p - q_h\|_P \;+\; \|\lambda - \beta_h\|_{\mathrm{I\!R}^m}\right)\,.
\end{aligned}
$$

Estimate (4.7) then follows using the triangle inequality.

■

Remark: As $\mathbf{u}_h$ exactly satisfies the specified flow rates, the error in the Lagrange multipliers does not appear in the error estimate for $\|\mathbf{u} - \mathbf{u}_h\|_X$.

Corollary 4.1 *For* $(\sigma\,,\,\mathbf{u}\,,\,p\,,\,\lambda) \in \left(W^{l+1,r'}\right)^{\acute{d}\times\acute{d}} \times \left(W^{k+1,r}\right)^{\acute{d}} \times W^{n+1,r'} \times \mathrm{I\!R}^m$ *satisfying (2.12)–(2.14) and* $(\sigma_h\,,\,\mathbf{u}_h\,,\,p_h\,,\,\lambda_h)$ *satisfying (3.6)–(3.8) (with* T_h*,* X_h*,* P_h *defined in (3.1)–(3.3)), for* h *sufficiently small, we have with* $\tilde{l} := \min\{(l+1)r/r'\,,\,k\,r/r'\,,\,n+1\}$ *that*

$$
\|\sigma - \sigma_h\|_T \;+\; \|\mathbf{u} - \mathbf{u}_h\|_X \;+\; \|p - q_h\|_P \;+\; \|\lambda - \beta_h\|_{\mathrm{I\!R}^m} \;\le\; C\,h^{\tilde{l}}\,. \tag{4.23}
$$

Proof: Estimate (4.26) follows from (4.3)–(4.7) and the approximating properties of continuous piecewise polynomials. (Note that, by assumption, $r \le r'$.)

■

5 Numerical Computations

In this section we present numerical results, obtained using MATLAB, for a flow problem subject (only) to specified flow rates conditions at the inflow and outflow boundaries. Along the other boundaries we impose the usual non-slip condition for the fluid velocity. In order to demonstrate the theoretical results derived in Section 4 we consider a simple model problem of flow in a square domain, $(0\,5) \times (0\,5)$, with inflow boundaries: $x = 0$, $1 < y < 2$, and $x = 0$, $3 < y < 4$, and outflow boundary at $x = 5$, $2 < y < 3$. The inflow rates were specified to be 4/3 and 2/3, respectively, with the outflow rate corresponding given as 2. (See Figure 5.1.)

Computations were performed on a sequence of four meshes, each mesh a uniform refinement (each triangle subdivided into four similar/smaller triangles) of the preceeding mesh. The second computational mesh is shown in Figure 5.2. The approximating nonlinear system was solved using a Newton method. For the approximation of the velocity and pressure we used continuous piecewise quadratic and continuous piecewise linear finite elements, respectively, (i.e. the Taylor-Hood pair). For the approximation of the stress we used continuous piecewise linear finite elements.

For the constitutive equation of the fluid we considered the power law equation (2.7), which, in the notation of (2.15), is rewritten as

$$
d(\mathbf{u}) \;=\; \nu_0^{1-r'}|\sigma|^{r'-2}\,\sigma \;=\; \breve{g}(\sigma)\sigma\;. \tag{5.1}
$$

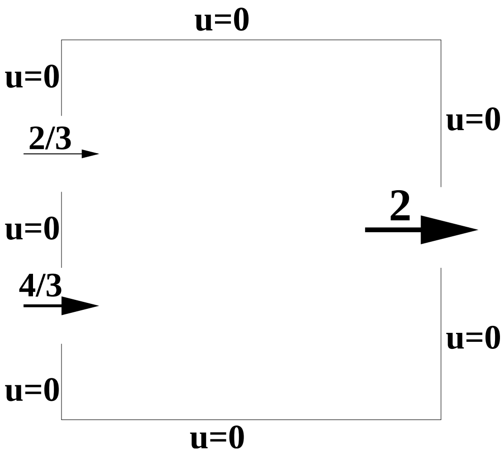

Figure 5.1: The Flow Problem.

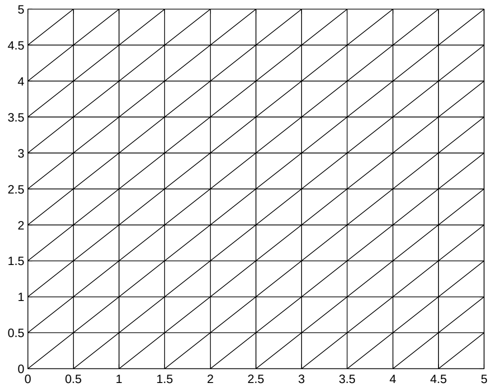

Figure 5.2: The second computational mesh, $h = 1/2$.

Presented in Table 5.1 and Figure 5.3 are the results of the computations for the parameter $r = 2$ ($r' = 2$), and in Table 5.2 and Figure 5.4 are the results of the computations for the parameter $r = 3/2$ ($r' = 3$).

	$\|\nabla \mathbf{u}_h\|_{L^2}$	$\|\nabla(\mathbf{u}_h - \mathbf{u}_{2h})\|_{L^2}$	$\tilde{\alpha}_{\mathbf{u}}$	$\|\sigma_h\|_{L^2}$	$\|\sigma_h - \sigma_{2h}\|_{L^2}$	$\tilde{\alpha}_\sigma$
$h = 1$	6.014			2.920		
$h = 1/2$	5.763	3.192		3.836	2.601	
$h = 1/4$	5.662	1.880	0.76	3.889	1.565	0.73
$h = 1/8$	5.614	1.213	0.63	3.902	1.026	0.61

Table 5.1: Norms of the velocity and stress for $r = 2$.

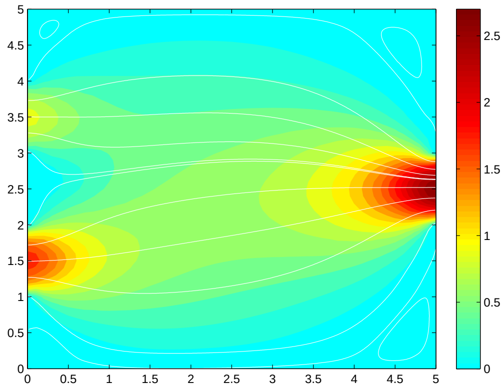

Figure 5.3: Plot of the magnitude of the velocity and streamlines for $r = 2$.

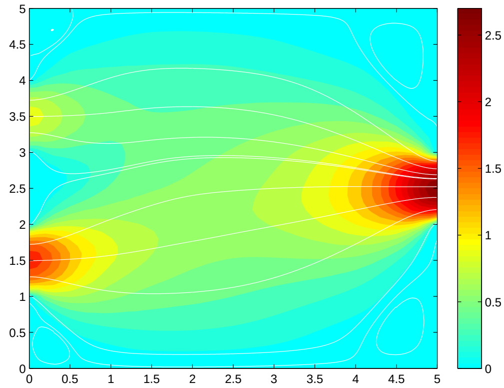

Figure 5.4: Plot of the magnitude of the velocity and streamlines for $r = 3/2$.

Assuming the convergence rate for the velocity is $\alpha_{\mathbf{u}}$, i.e. $\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L^r} \sim Ch^{\alpha_{\mathbf{u}}}$, we compute the

	$\|\nabla \mathbf{u}_h\|_{L^{3/2}}$	$\|\nabla(\mathbf{u}_h - \mathbf{u}_{2h})\|_{L^{3/2}}$	$\tilde{\alpha}_{\mathbf{u}}$	$\|\sigma_h\|_{L^3}$	$\|\sigma_h - \sigma_{2h}\|_{L^3}$	$\tilde{\alpha}_\sigma$
$h = 1$	8.680			1.758		
$h = 1/2$	8.664	4.710		1.995	1.279	
$h = 1/4$	8.531	2.607	0.85	2.063	0.871	0.55
$h = 1/8$	8.526	1.469	0.83	2.249	0.697	0.32

Table 5.2: Norms of the velocity and stress for $r = 3/2$.

experimental convergence rate for the velocity using

$$\|\nabla(\mathbf{u}_h - \mathbf{u}_{2h})\|_{L^r} \;\leq\; \|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L^r} + \|\nabla(\mathbf{u} - \mathbf{u}_{2h})\|_{L^r} \;\sim\; \tilde{C} h^{\alpha_{\mathbf{u}}} .$$

Therefore
$$\tilde{\alpha}_{\mathbf{u}} = \log\left(\|\nabla(\mathbf{u}_h - \mathbf{u}_{2h})\|_{L^r} / \|\nabla(\mathbf{u}_{2h} - \mathbf{u}_{4h})\|_{L^r}\right) / \log(2) . \quad (5.2)$$

Similarly
$$\tilde{\alpha}_\sigma = \log\left(\|\sigma_h - \sigma_{2h}\|_{L^{r'}} / \|\sigma_{2h} - \sigma_{4h}\|_{L^{r'}}\right) / \log(2) . \quad (5.3)$$

The case $r = 2$ $(r' = 2)$

For the case $r = 2$ $(r' = 2)$ the constitutive equation describes a "Newtonian" fluid and the problem becomes a (linear) three-field Stokes problem with defective boundary conditions. As in this case $\sigma = \nu_0 d(\mathbf{u}) = \nu_0/2\,(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$, and we are constructing a piecewise linear approximation for the σ and a piecewise quadratic approximation for $\mathbf{u}$, we expect that $\tilde{\alpha}_{\mathbf{u}} \approx \tilde{\alpha}_\sigma$, as observed in Table 5.1. The fact that $\tilde{\alpha}_{\mathbf{u}} \approx \tilde{\alpha}_\sigma \neq 2$ is due to the lack of regularity of $\mathbf{u}$ and σ, attributable to the singular behavior of $\nabla\mathbf{u}$ and σ at the corners of the inflow and outflow boundaries.

The case $r = 3/2$ $(r' = 3)$

Note that for this case the velocity is in $\left(W^{1,3/2}(\Omega)\right)^2$ and the stress in $\left(L^3(\Omega)\right)^4_{sym}$. Also, we have that $\sigma = \nu_0 |d(\mathbf{u})|^{-0.5} d(\mathbf{u})$.

The a priori error estimates presented in Theorem 4.1 are dominated by the term $\|\sigma - \tau_h\|_T^r$ on the RHS of (4.3). If this term was not present, the a priori estimate would represent the *best approximation error* (for appropriately chosen approximation spaces for σ_h, $\mathbf{u}_h$, p_h). The computations in Table 5.2 are consistent with the approximations being *best approximations* (see below). This may be due to the fact that the behavior of the computational results are pre-asymptotic, or that the estimates in Theorem 4.1 are not optimal.

At the endpoints of the inflow/outflow boundaries $\nabla\mathbf{u}$ will be singular. Assuming that at these points $\nabla\mathbf{u}$ has a point singularity of the form ρ^{-s}, $0 < s < 1$, where ρ denotes the distance from the singular point; $\mathbf{u}_I$ is a continuous piecewise quadratic interpolant of $\mathbf{u}$, then we expect that

$$\begin{aligned}
\|\nabla(\mathbf{u} - \mathbf{u}_I)\|_{L^r} &\sim \left(\int_{B(0,h)} (\rho^{-s})^r \, dA \;+\; \int_{B(0,R)\setminus B(0,h)} (h^2\,\rho^{-s-2})^r \, dA\right)^{1/r} \\
&= \left(\int_{\theta=0}^{\pi}\int_{\rho=0}^{h} \rho\,\rho^{-sr}\, d\rho\, d\theta \;+\; \int_{\theta=0}^{\pi}\int_{\rho=h}^{R} \rho\, h^{2r} \rho^{-(s+2)r}\, d\rho\, d\theta\right)^{1/r} \\
&\sim C\, h^{(2-rs)/r}
\end{aligned}$$

$$\text{i.e.} \quad \alpha_{\mathbf{u}} = (2 - rs)/r . \quad (5.4)$$

From (2.7), and for σ_I a continuous piecewise linear interpolant of σ, we would expect

$$\begin{aligned}
\|\sigma - \sigma_I\|_{L^{r'}} &\sim \left(\int_{B(0,h)} \left((\rho^{-s})^{r-2} \rho^{-s} \right)^{r'} dA \; + \; \int_{B(0,R)\setminus B(0,h)} \left(h^2 \, (\rho^{-s})^{r-2} \rho^{-s-2}) \right)^{r'} dA \right)^{1/r'} \\
&\sim C\, h^{(2-rs)/r'} \\
\text{i.e.} \quad \alpha_\sigma &= (2 - rs)/r' \, . \qquad (5.5)
\end{aligned}$$

For $r = 3/2$, $r' = 3$, from (5.4),(5.5) we have that $\alpha_{\mathbf{u}}/\alpha_\sigma = r'/r = 2$, which is consistent with the computations in Table 5.2.

Comparing Figures 5.3 and 5.4, the flow fields corresponding to $r = 2$ and $r = 3/2$, respectively, we observe (as expected) the larger vortices in the upper and lower right hand corners of Ω for the case $r = 3/2$. The magnitude of $\mathbf{u}(x, y) = [u_1(x, y) \, , \, u_2(x, y)]$ plotted in Figures 5.3 and 5.4 was calculated via $|\mathbf{u}(x, y)| = (u_1(x, y)^r + u_2(x, y)^r)^{1/r}$.

For comparison, in Figure 5.5 is the flow field for the case $r = 3/2$ where we specify parabolic velocity inflow profiles (with inflow rates 4/3 and 2/3 respectively) and a "do nothing" (i.e. $\sigma - pI = \mathbf{0}$) outflow boundary condition. The flow field looks very similar to the that in Figure 5.4 where the flow rates were imposed using the Lagrange multiplier approach. The values for the velocity semi-norm and the stress norm in Figure 5.5 are 8.718 and 2.081, compared to 8.531 and 2.063 in Figure 5.4.

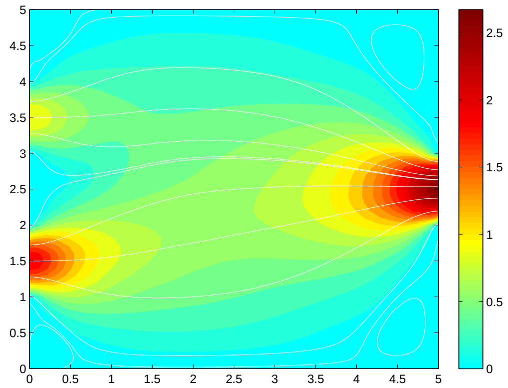

Figure 5.5: Plot of the magnitude of the velocity and streamlines for $r = 3/2$ with parabolic inflow boundary conditions and a "do nothing" outflow boundary condition.

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