A domain decomposition method for the Oseen-viscoelastic flow equations

Eleanor Jenkins * Hyesuk Lee *

Abstract

We study a non-overlapping domain decomposition method for the Oseen-viscoelastic flow problem. The data on the interface are transported through Newmann and Dirichlet boundary conditions for the momentum and constitutive equations, respectively. The discrete variational formulations of subproblems are presented and investigated for the existence of solutions. We show convergence of the domain decomposition solution to a solution of the one-domain problem. Convergence of an iterative algorithm and some numerical results are also presented.

Key words domain decomposition, viscoelastic flows, finite element method

1 Introduction

In this paper we investigate a domain decomposition method for the Oseen-viscoelastic fluid flow of Johnson-Segalman type. Viscoelastic flow equations consist of a constitutive equation for the material, a momentum equation, and a mass equation. For Newtonian fluids, the equations for stationary, incompressible, creeping flow can be simplified in terms of velocity and pressure because of the simplistic relationship between stress and velocity. For non-Newtonian fluids, however, there is an "extra" stress component that accounts for the forces that the material develops under deformation. Thus, no simplification takes place, and the complete system consists of three equations in the unknowns pressure (scalar), velocity (vector), and stress (symmetric tensor). One of the difficulties in simulating viscoelastic flows arises from the hyperbolic nature of the constitutive equation for which one needs to use a stabilization technique such as the streamline upwinding Petrov-Galerkin (SUPG) method and the discontinuous Galerkin method [1].

Here we consider the discontinuous Galerkin method for the finite element approximation of stress. When discretized using a discontinuous space to approximate the stress tensor, the size of the discrete system is increased considerably increased. In order to handle the large number of unknowns associated with the viscoelastic systems, we propose a domain decomposition algorithm for a two-subdomain problem. An extension to a muti-domain problem is straightforward. In each subdomain we impose a Neumann condition for the momentum equation and a Dirichlet condition for the constitutive equation on the interface boundary in order to transport data. The idea of using a Neumann type condition is based on the methods found in [15, 17, 18]. For the finite element approximation, we use the standard Taylor-Hood element for velocity and pressure and the discontinuous Galerkin method for stress.

Domain decomposition methods have been used extensively to solve elliptic partial differential equations (see [22] and references therein), along with the Stokes and Navier-Stokes equations [8, 10, 13, 15, 17]. Some domain decomposition methods for viscoelastic fluids are found in [4, 16, 23]. In [4] the authors use the additive Schwarz method in conjunction with the DEVSS-G operator

*Department of Mathematical Sciences, Clemson University, Clemson, SC 29634-0975, USA, lea@clemson.edu hklee@clemson.edu. This work was partially supported by the NSF under grant no. DMS-0410792.

splitting method. The two-level Schwarz method is applied to a Stokes-like problem that results from the splitting. A domain decomposition spectral collocation (DDSC) method is introduced in [23] for an Oldroyd-B fluid in model porous geometries. The method discussed in [16] is based on the concepts of streamlines and local transformation functions, where the local functions are the primary unknowns. The functions map the physical subdomains to transformed domains where the mapped streamlines are parallel straight lines.

The remainder of the document is organized as follows. In Section 2, we introduce the model equations for the Oseen-viscoelastic flow problem, and in Section 3, we describe the model equations for the problem decomposed into two subdomains. Convergence of a domain decomposition solution to a solution of the one-domain problem is discussed in Section 4. In Section 5, we present iterative algorithms and prove convergence of the algorithms. We provide numerical results in Section 6, and conclusions and future work are given in Section 7.

2 Model equations and finite element approximation

Let Ω be a bounded domain in $\mathbb{R}^d$ with the Lipschitz continuous boundary $\partial\Omega$. Consider the Johnson-Segalman problem

$$\boldsymbol{\sigma} + \lambda(\mathbf{u}\cdot\nabla)\boldsymbol{\sigma} + \lambda g_a(\boldsymbol{\sigma}, \nabla\mathbf{u}) - 2\,\alpha\,\mathbf{D}(\mathbf{u}) = \mathbf{0} \quad \text{in } \Omega\,, \tag{2.1}$$

$$-\nabla\cdot\boldsymbol{\sigma} - 2(1-\alpha)\,\nabla\cdot D(\mathbf{u}) + \nabla p = \mathbf{f} \quad \text{in } \Omega\,, \tag{2.2}$$

$$div\,\mathbf{u} = 0 \quad \text{in } \Omega\,, \tag{2.3}$$

where $\boldsymbol{\sigma}$ denotes the polymeric stress tensor, $\mathbf{u}$ the velocity vector, p the pressure of fluid, and λ is the Weissenberg number defined as the product of the relaxation time and a characteristic strain rate. Assume that p has zero mean value over Ω. In (2.1) and (2.2), $\mathbf{D}(\mathbf{u}) := (\nabla\mathbf{u} + \nabla\mathbf{u}^T)/2$ is the rate of the strain tensor, α a number such that $0 < \alpha < 1$ which may be considered as the fraction of viscoelastic viscosity, and $\mathbf{f}$ the body force. In (2.1), $g_a(\boldsymbol{\sigma}, \nabla\mathbf{u})$ is defined by

$$g_a(\boldsymbol{\sigma}, \nabla\mathbf{u}) := \frac{1-a}{2}(\boldsymbol{\sigma}\nabla\mathbf{u} + \nabla\mathbf{u}^T\boldsymbol{\sigma}) - \frac{1+a}{2}(\nabla\mathbf{u}\,\boldsymbol{\sigma} + \boldsymbol{\sigma}\nabla\mathbf{u}^T) \tag{2.4}$$

for $a \in [-1, 1]$.

We use the Sobolev spaces $W^{m,p}(D)$ with norms $\|\cdot\|_{m,p,D}$ if $p < \infty$, $\|\cdot\|_{m,\infty,D}$ if $p = \infty$. We denote the Sobolev space $W^{m,2}$ by H^m with the norm $\|\cdot\|_m$. The corresponding space of vector-valued or tensor-valued functions is denoted by $\mathbf{H}^m$. If $D = \Omega$, D is omitted, i.e., $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$ and $\|\cdot\| = \|\cdot\|_\Omega$. Existence of a solution to the problem (2.1)-(2.3) with the homogeneous boundary condition for $\mathbf{u}$ has been documented by Renardy ([19]) with the small data condition: if $\mathbf{f}$ is sufficiently regular and small, the problem admits a unique bounded solution $(\mathbf{u}, p, \boldsymbol{\sigma}) \in \mathbf{H}^3(\Omega) \times H^2(\Omega) \times \mathbf{H}^2(\Omega)$.

In this paper, the constitutive equation (2.1) is simplified to define our domain decomposition problem. We will consider the linear Oseen problem with the given velocity $\mathbf{b}(\mathbf{x})$:

$$\boldsymbol{\sigma} + \lambda(\mathbf{b}\cdot\nabla)\boldsymbol{\sigma} + \lambda g_a(\boldsymbol{\sigma}, \nabla\mathbf{b}) - 2\,\alpha\,\mathbf{D}(\mathbf{u}) = \mathbf{0} \quad \text{in } \Omega\,, \tag{2.5}$$

$$-\nabla\cdot\boldsymbol{\sigma} - 2(1-\alpha)\,\nabla\cdot\mathbf{D}(\mathbf{u}) + \nabla p = \mathbf{f} \quad \text{in } \Omega\,, \tag{2.6}$$

$$div\,\mathbf{u} = 0 \quad \text{in } \Omega\,, \tag{2.7}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega\,. \tag{2.8}$$

We make the following assumption for $\mathbf{b}$:

$$\mathbf{b} \in \mathbf{H}^1_0(\Omega), \qquad \nabla\cdot\mathbf{b} = 0, \qquad \|\mathbf{b}\|_\infty \le M, \qquad \|\nabla\mathbf{b}\|_\infty \le M < \infty.$$

Define the function spaces for the velocity $\mathbf{u}$, the pressure p and the stress $\boldsymbol{\sigma}$, respectively:

$$\mathbf{X} := \mathbf{H}_0^1(\Omega)\,,$$
$$S := L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q\, d\Omega = 0\}\,,$$
$$\boldsymbol{\Sigma} := \{\boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \tau_{ij} = \tau_{ji},\ (\mathbf{b}\cdot\nabla)\boldsymbol{\tau} \in \mathbf{L}^2(\Omega)\}\,.$$

Note that the velocity and pressure spaces, $\mathbf{X}$ and S, satisfy the *inf-sup* condition

$$\inf_{q\in S}\sup_{\mathbf{v}\in\mathbf{X}} \frac{(q, \nabla\cdot\mathbf{v})}{\|\mathbf{v}\|_1\|q\|_0} \geq C\,. \tag{2.9}$$

The corresponding weak formulation is then given by:

$$(\boldsymbol{\sigma},\boldsymbol{\tau}) + \lambda((\mathbf{b}\cdot\nabla)\boldsymbol{\sigma},\boldsymbol{\tau}) + \lambda(g_a(\boldsymbol{\sigma},\nabla\mathbf{b}),\boldsymbol{\tau}) - 2\alpha(\mathbf{D}(\mathbf{u}),\boldsymbol{\tau}) = 0 \quad \forall\boldsymbol{\tau}\in\boldsymbol{\Sigma}\,, \tag{2.10}$$

$$(\boldsymbol{\sigma},\mathbf{D}(\mathbf{v})) + 2(1-\alpha)(\mathbf{D}(\mathbf{u}),\mathbf{D}(\mathbf{v})) - (p,\nabla\cdot\mathbf{v}) = (\mathbf{f},\mathbf{v}) \quad \forall\mathbf{v}\in\mathbf{X}\,, \tag{2.11}$$

$$(q,\nabla\cdot\mathbf{u}) = 0 \quad \forall q\in S\,. \tag{2.12}$$

In the weak divergence free space

$$\boldsymbol{V} := \{\mathbf{v}\in\mathbf{X} : \int_\Omega q\, div\,\mathbf{v}\, d\Omega = 0 \quad \forall q \in L_0^2(\Omega)\}\,,$$

the weak formulation (2.10)-(2.12) is equivalent to:

$$(\boldsymbol{\sigma},\boldsymbol{\tau}) + \lambda((\mathbf{b}\cdot\nabla)\boldsymbol{\sigma},\boldsymbol{\tau}) + \lambda(g_a(\boldsymbol{\sigma},\nabla\mathbf{b}),\boldsymbol{\tau}) - 2\alpha(\mathbf{D}(\mathbf{u}),\boldsymbol{\tau}) = 0 \quad \forall\boldsymbol{\tau}\in\boldsymbol{\Sigma}\,, \tag{2.13}$$

$$(\boldsymbol{\sigma}, d(\mathbf{v})) + 2(1-\alpha)(\mathbf{D}(\mathbf{u}),\mathbf{D}(\mathbf{v})) = (\mathbf{f},\mathbf{v}) \quad \forall\mathbf{v}\in\boldsymbol{V}\,. \tag{2.14}$$

We now consider the finite element approximation of the problem (2.10)-(2.12). Suppose T^h is a triangulation of Ω such that $\overline{\Omega} = \{\cup K : K \in T^h\}$. Assume that there exist positive constants c_1, c_2 such that

$$c_1 h \leq h_K \leq c_2\rho_K\,,$$

where h_K is the diameter of K, ρ_K is the diameter of the greatest ball included in K, and $h = \max_{K\in T^h} h_K$.

Due to the hyperbolic nature of the constitutive equation, a stabilization technique is needed for the finite element simulation of viscoelastic flows. Streamline upwinding ([14], [20]) and the discontinuous Galerkin method ([2], [7]) are the commonly used discretization techniques to handle this problem. We use the discontinuous Galerkin method for approximating the stress. Let $P_k(K)$ denote the space of polynomials of degree less than or equal to k on $K\in T^h$. Then we define finite element spaces for the approximation of $(\mathbf{u}, p)$:

$$\mathbf{X}^h := \{\mathbf{v}\in\mathbf{X}\cap(C^0(\overline{\Omega}))^d : \mathbf{v}|_K \in P_2(K)^d,\ \forall K\in T^h\}\,,$$
$$S^h := \{q\in S\cap C^0(\overline{\Omega}) : q|_K \in P_1(K),\ \forall K\in T^h\}\,,$$
$$\boldsymbol{V}^h := \{\mathbf{v}\in\mathbf{X}^h : (q,\nabla\cdot\mathbf{v}) = 0,\ \forall q\in S^h\}\,.$$

The stress $\boldsymbol{\sigma}$ is approximated in the discontinuous finite element space of piecewise linears:

$$\boldsymbol{\Sigma}^h := \{\boldsymbol{\tau}\in\boldsymbol{\Sigma} : \boldsymbol{\tau}|_K \in P_1(K)^{d\times d},\ \forall K\in T^h\}\,.$$

We introduce some notation below in order to analyze an approximate solution by the discontinuous Galerkin method. We define

$$\partial K^-(\mathbf{b}) := \{\mathbf{x} \in \partial K,\ \mathbf{b}\cdot\mathbf{n} < 0\}\,,$$

where ∂K is the boundary of K and $\mathbf{n}$ is outward unit normal, and

$$\boldsymbol{\tau}^{\pm}(\mathbf{b}) := \lim_{\epsilon\longrightarrow 0} \boldsymbol{\tau}(\mathbf{x} \pm \epsilon\mathbf{b}(\mathbf{x}))\,.$$

We also define

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_h := \sum_{K\in T^h} (\boldsymbol{\sigma}, \boldsymbol{\tau})_K\,,$$

$$< \boldsymbol{\sigma}^{\pm}, \boldsymbol{\tau}^{\pm} >_{h,\mathbf{b}} := \sum_{K\in T^h} \int_{\partial K^-(\mathbf{b})} (\boldsymbol{\sigma}^{\pm}(\mathbf{b}), \boldsymbol{\tau}^{\pm}(\mathbf{b}))|\mathbf{n}\cdot\mathbf{b}|\, ds\,.$$

$$\|\boldsymbol{\tau}\|_{0,\Gamma^h} := \left(\sum_{K\in T^h} |\boldsymbol{\tau}|^2_{0,\partial K}\right)^{1/2}$$

for $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \prod_{K\in T^h} (L^2(K))^{d\times d}$, and

$$\|\boldsymbol{\tau}\|_{m,h} := \left(\sum_{K\in T^h} |\boldsymbol{\tau}|^2_{m,K}\right)^{1/2}$$

for $\boldsymbol{\tau} \in \prod_{K\in T^h} (W^{m,2}(K))^{d\times d}$, if $m < \infty$.

The discontinuous Galerkin finite element approximation of (2.10)–(2.12) is then as follows: *find* $\mathbf{u}^h \in \mathbf{X}^h$, $p^h \in S^h$, $\boldsymbol{\sigma}^h \in \boldsymbol{\Sigma}^h$ *such that*

$$(\boldsymbol{\sigma}^h, \boldsymbol{\tau}^h) + \lambda\left[((\mathbf{b}\cdot\nabla)\boldsymbol{\sigma}^h, \boldsymbol{\tau}^h)_h + < \boldsymbol{\sigma}^{h+} - \boldsymbol{\sigma}^{h-}, \boldsymbol{\tau}^{h+} >_{h,\mathbf{b}}\right] \tag{2.15}$$

$$+\lambda(g_a(\boldsymbol{\sigma}^h, \nabla\mathbf{b}^h), \boldsymbol{\tau}^h) - 2\alpha(d(\mathbf{u}^h), \boldsymbol{\tau}^h) = 0 \quad \forall \boldsymbol{\tau}^h \in \boldsymbol{\Sigma}^h\,,$$

$$(\boldsymbol{\sigma}^h, d(\mathbf{v}^h)) + 2(1-\alpha)\,(d(\mathbf{u}^h), d(\mathbf{v}^h)) - (p^h, \nabla\cdot\mathbf{v}^h) = (\mathbf{f}, \mathbf{v}^h) \quad \forall \mathbf{v}^h \in \mathbf{X}^h\,, \tag{2.16}$$

$$(q^h, \nabla\cdot\mathbf{u}^h) = 0 \quad \forall q^h \in S^h\,. \tag{2.17}$$

It was shown in [6] that the discrete problem (2.15)-(2.17) has a unique solution if $1 - 2\lambda M d > 0$, and the solution satisfies the error estimate

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^h\|_0 + \|\mathbf{u} - \mathbf{u}^h\|_1 \le Ch. \tag{2.18}$$

.

3 Description of the domain decomposition method

We describe the domain decomposition method using two subdomains, and the analysis that follows is based on this problem. However, there are no difficulties associated with extending the analysis to handle as many subdomains as necessary.

Let Ω be divided into two disjoint subdomains Ω_1 and Ω_2 such that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$. The *interface* between the two domains is denoted by Γ_0 so that $\Gamma_0 = \overline{\Omega}_1 \cap \overline{\Omega}_2$. Define

$$\Gamma^-_{12} := \{\mathbf{x} \in \Gamma_0 : \mathbf{b}\cdot\mathbf{n}_1 < 0\}, \qquad \Gamma^-_{21} := \{\mathbf{x} \in \Gamma_0 : \mathbf{b}\cdot\mathbf{n}_2 < 0\},$$

where $\mathbf{n}_i$ is the outward unit normal vector to Ω_i for $i = 1, 2$. Let $\Gamma_1 = \overline{\Omega}_1 \cap \partial\Omega$ and $\Gamma_2 = \overline{\Omega}_2 \cap \partial\Omega$.

We consider the following pair of Oseen-viscoelastic systems with mixed boundary conditions:

$$\boldsymbol{\sigma}_1 + \lambda(\mathbf{b}\cdot\nabla)\boldsymbol{\sigma}_1 + \lambda g_a(\boldsymbol{\sigma}_1, \nabla\mathbf{b}) - 2\,\alpha\,\mathbf{D}(\mathbf{u}_1) = \mathbf{0} \quad \text{in } \Omega_1\,, \tag{3.1}$$

$$-\nabla\cdot\boldsymbol{\sigma}_1 - 2(1-\alpha)\,\nabla\cdot\mathbf{D}(\mathbf{u}_1) + \nabla p_1 = \mathbf{f} \quad \text{in } \Omega_1\,, \tag{3.2}$$

$$div\,\mathbf{u}_1 = 0 \quad \text{in } \Omega_1\,, \tag{3.3}$$

$$\mathbf{u}_1 = \mathbf{0} \quad \text{on } \Gamma_1\,, \tag{3.4}$$

$$(\boldsymbol{\sigma}_1 + 2(1-\alpha)\,\mathbf{D}(\mathbf{u}_1) - p_1\,I)\cdot\mathbf{n}_1 = -\frac{1}{\epsilon}(\mathbf{u}_1 - \mathbf{u}_2) \quad \text{on } \Gamma_0\,, \tag{3.5}$$

$$\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2 \quad \text{on } \Gamma_{12}^-\,, \tag{3.6}$$

and

$$\boldsymbol{\sigma}_2 + \lambda(\mathbf{b}\cdot\nabla)\boldsymbol{\sigma}_2 + \lambda g_a(\boldsymbol{\sigma}_2, \nabla\mathbf{b}) - 2\,\alpha\,\mathbf{D}(\mathbf{u}_2) = \mathbf{0} \quad \text{in } \Omega_2\,, \tag{3.7}$$

$$-\nabla\cdot\boldsymbol{\sigma}_1 - 2(1-\alpha)\,\nabla\cdot\mathbf{D}(\mathbf{u}_2) + \nabla p_2 = \mathbf{f} \quad \text{in } \Omega_2\,, \tag{3.8}$$

$$div\,\mathbf{u}_2 = 0 \quad \text{in } \Omega_2\,, \tag{3.9}$$

$$\mathbf{u}_2 = \mathbf{0} \quad \text{on } \Gamma_2\,, \tag{3.10}$$

$$(\boldsymbol{\sigma}_2 + 2(1-\alpha)\,\mathbf{D}(\mathbf{u}_2) - p_2\,I)\cdot\mathbf{n}_2 = \frac{1}{\epsilon}(\mathbf{u}_1 - \mathbf{u}_2) \quad \text{on } \Gamma_0\,, \tag{3.11}$$

$$\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_1 \quad \text{on } \Gamma_{21}^-\,. \tag{3.12}$$

Note that (3.6), the Dirichlet boundary conditions for $\boldsymbol{\sigma}_1$, is imposed on a part of the interface Γ_0 where $\mathbf{b}\cdot\mathbf{n}_1 < 0$. Similarly, (3.12) is imposed on the inflow boundary of Ω_2 associated with the given velocity field $\mathbf{b}$.

Define the function spaces for the velocity $\mathbf{u}_i$, the pressure p_i and the stress $\boldsymbol{\sigma}_i$, respectively, for $i = 1, 2$:

$$\mathbf{X}_i := \{\mathbf{v} \in \mathbf{H}^1(\Omega_i) :\ \mathbf{v} = \mathbf{0} \quad \text{on } \Gamma_i\}\,,$$
$$S_i := L^2(\Omega_i)\,,$$
$$\boldsymbol{\Sigma}_i := (L^2(\Omega_i))^{2\times 2} \cap \{\boldsymbol{\tau} = (\tau_{ij}) : \tau_{ij} = \tau_{ji},\ \mathbf{b}\cdot\nabla\boldsymbol{\tau} \in (L^2(\Omega_i))^{2\times 2}\}$$

and the *div free* space

$$\boldsymbol{V}_i := \{\mathbf{v} \in \mathbf{X}_i : \int_{\Omega_i} q\,div\,\mathbf{v}\,d\Omega_i = 0 \quad \forall q \in S_i\}\,.$$

The corresponding weak formulation of (3.1)-(3.12) is then given by

$$(\boldsymbol{\sigma}_1, \boldsymbol{\tau}_1) + \lambda\,(\mathbf{b}\cdot\nabla)\boldsymbol{\sigma}_1, \boldsymbol{\tau}_1) + \lambda\,(g_a(\boldsymbol{\sigma}_1, \nabla\mathbf{b}), \boldsymbol{\tau}_1) - 2\alpha(\mathbf{D}(\mathbf{u}_1), \boldsymbol{\tau}_1) = \mathbf{0} \quad \forall\boldsymbol{\tau}_1 \in \boldsymbol{\Sigma}_1\,, \tag{3.13}$$

$$(\boldsymbol{\sigma}_1, \mathbf{D}(\mathbf{v}_1)) + 2(1-\alpha)(\mathbf{D}(\mathbf{u}_1), \mathbf{D}(\mathbf{v}_1)) - (p_1, \nabla\cdot\mathbf{v}_1) + \frac{1}{\epsilon}(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{v}_1)_{\Gamma_0} = (\mathbf{f}, \mathbf{v}_1) \quad \forall\mathbf{v}_1 \in \mathbf{X}_1\,, \tag{3.14}$$

$$(q_1, \nabla\cdot\mathbf{u}_1) = 0 \quad \forall q_1 \in S_1\,, \tag{3.15}$$

$$\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2 \quad \text{on } \Gamma_{12}^-\,, \tag{3.16}$$

and

$$(\boldsymbol{\sigma}_2, \boldsymbol{\tau}_2) + \lambda\,(\mathbf{b}\cdot\nabla)\boldsymbol{\sigma}_2, \boldsymbol{\tau}_2) + \lambda\,(g_a(\boldsymbol{\sigma}_2, \nabla\mathbf{b}), \boldsymbol{\tau}_2) - 2\alpha(\mathbf{D}(\mathbf{u}_2), \boldsymbol{\tau}_2) = \mathbf{0} \quad \forall \boldsymbol{\tau}_2 \in \boldsymbol{\Sigma}_2\,, \tag{3.17}$$

$$(\boldsymbol{\sigma}_2, \mathbf{D}(\mathbf{v}_2)) + 2(1-\alpha)(\mathbf{D}(\mathbf{u}_2), \mathbf{D}(\mathbf{v}_2)) - (p_2, \nabla\cdot\mathbf{v}_2) - \frac{1}{\epsilon}(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{v}_2)_{\Gamma_0} = (\mathbf{f}, \mathbf{v}_2) \quad \forall \mathbf{v}_2 \in \mathbf{X}_2\,, \tag{3.18}$$

$$(q_2, \nabla\cdot\mathbf{u}_2) = 0 \quad \forall q_2 \in S_2\,. \tag{3.19}$$

$$\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_1 \quad \text{on } \Gamma_{21}^-\,. \tag{3.20}$$

We define discrete subspaces for the finite element approximation of the two-domain problem (3.13)-(3.20). Let T_i^h denote a triangulation of Ω_i such that $\overline{\Omega}_i = \{\cup K : K \in T_i^h\}$ for $i = 1, 2$. Define finite element spaces for the approximation of $(\boldsymbol{\sigma}_i^h, \mathbf{u}_i^h, p_i^h)$ for $i = 1, 2$:

$$\begin{aligned}
\boldsymbol{\Sigma}_i^h &:= \{\boldsymbol{\tau} \in \boldsymbol{\Sigma}_i : \boldsymbol{\tau}|_K \in P_1(K)^{d\times d},\ \forall K \in T_i^h\}\,,\\
\mathbf{X}_i^h &:= \{\mathbf{v} \in \mathbf{X}_i \cap (C^0(\overline{\Omega}_i))^d : \mathbf{v}|_K \in P_2(K)^d,\ \forall K \in T_i^h\}\,,\\
S_i^h &:= \{q \in S_i \cap C^0(\overline{\Omega}_i) : q|_K \in P_1(K),\ \forall K \in T_i^h\}\,.
\end{aligned}$$

We also define the discrete divergence free subspace

$$\boldsymbol{V}_i^h := \{\mathbf{v} \in \mathbf{X}_i^h : (q, \nabla\cdot\mathbf{v}) = 0,\ \forall q \in S_i^h\}\,.$$

The finite element spaces defined above satisfy the standard approximation properties (see [3] or [9]), i.e., there exist an integer k and a constant C such that

$$\inf_{\mathbf{v}^h\in\mathbf{X}_i^h} \|\mathbf{v} - \mathbf{v}^h\|_1 \le Ch^2\|\mathbf{v}\|_3 \quad \forall\,\mathbf{v} \in \mathbf{H}^3(\Omega_i)\,, \tag{3.21}$$

$$\inf_{q^h\in S_i^h} \|q - q^h\|_0 \le Ch^2\|q\|_2 \quad \forall\,q \in H^2(\Omega_i)\,, \tag{3.22}$$

and

$$\inf_{\boldsymbol{\tau}^h\in\Sigma_i^h} \|\boldsymbol{\tau} - \boldsymbol{\tau}^h\|_0 \le Ch^2\|\boldsymbol{\tau}\|_2 \quad \forall\,\boldsymbol{\tau} \in \mathbf{H}^2(\Omega_i)\,. \tag{3.23}$$

It is also well known that the Taylor-Hood pair $(\mathbf{X}_i^h, S_i^h)$ satisfies the *inf-sup* condition,

$$\inf_{0\neq q^h\in S_i^h}\ \sup_{0\neq\mathbf{v}^h\in\mathbf{X}_i^h} \frac{(q^h, \nabla\cdot\mathbf{v}^h)}{\|\mathbf{v}^h\|_1\|q^h\|_0} \ge C\,, \tag{3.24}$$

where C is a positive constant independent of h.

In addition to notation introduced in the previous section for the discontinuous Galerkin method, we present more notation to analyze $\boldsymbol{\sigma}_i^h$ on the interface Γ_0. We define, for $\boldsymbol{\sigma}_i, \boldsymbol{\tau}_i \in \prod_{K\in T_i^h} (L^2(K))^{d\times d}$

$$<\boldsymbol{\sigma}_i^\pm, \boldsymbol{\tau}_i^\pm>_{h,\mathbf{b}} := \sum_{K\in T_i^h, \partial K\cap\Gamma_0=\emptyset} \int_{\partial K_-(\mathbf{b})} (\boldsymbol{\sigma}_i^\pm(\mathbf{b}) : \boldsymbol{\tau}_i^\pm(\mathbf{b}))|\mathbf{n}\cdot\mathbf{b}|\,ds\,,$$

$$\ll \boldsymbol{\sigma}_i^\pm \gg_{h,\mathbf{b}} := <\boldsymbol{\sigma}_i^\pm, \boldsymbol{\sigma}_i^\pm>_{h,\mathbf{b}}^{1/2}\,,$$

$$\|\boldsymbol{\tau}_i\|_{0,\Gamma_i^h} := \left(\sum_{K\in T_i^h} |\boldsymbol{\tau}_i|_{0,\partial K}^2\right)^{1/2}\,,$$

and

$$\|\boldsymbol{\tau}_i\|_{m,h} := \left(\sum_{K\in T_i^h} |\boldsymbol{\tau}_i|^2_{m,K} \right)^{1/2}$$

for $\boldsymbol{\tau}_i \in \prod_{K\in T_i^h} (W^{m,2}(K))^{d\times d}$, if $m < \infty$. Also define, for $i,j = 1,2$

$$< \boldsymbol{\sigma}_i^{\pm}, \boldsymbol{\tau}_j^{\pm} >_{h,\mathbf{b},\Gamma_{lm}^-} := \sum_{K\in T_l^h, \partial K\cap \Gamma_{lm}^- \neq \emptyset} \int_{\partial K_-(\mathbf{b})} (\boldsymbol{\sigma}_i^{\pm}(\mathbf{b}) : \boldsymbol{\tau}_j^{\pm}(\mathbf{b})) |\mathbf{n}\cdot\mathbf{b}|\, ds\,,$$

$$\ll \boldsymbol{\sigma}_i^{\pm} \gg_{h,\mathbf{b},\Gamma_{lm}^-} :=< \boldsymbol{\sigma}_i^{\pm}, \boldsymbol{\sigma}_i^{\pm} >^{1/2}_{h,\mathbf{b},\Gamma_{lm}^-}\,,$$

where $(l,m) = (1,2)$ or $(2,1)$. The discrete variational formulations of (3.13)-(3.16) and (3.17)-(3.20) are then given by

$$\begin{aligned} &(\boldsymbol{\sigma}_1^h, \boldsymbol{\tau}_1^h) + \lambda\,((\mathbf{b}\cdot\nabla)\boldsymbol{\sigma}_1^h, \boldsymbol{\tau}_1^h)_h + \lambda\,(g_a(\boldsymbol{\sigma}_1^h, \nabla\mathbf{b}), \boldsymbol{\tau}_1^h) - 2\alpha(\mathbf{D}(\mathbf{u}_1^h), \boldsymbol{\tau}_1^h) \\ &\quad +\lambda\, < {\boldsymbol{\sigma}_1^h}^+ - {\boldsymbol{\sigma}_1^h}^-, {\boldsymbol{\tau}_1^h}^+ >_{h,\mathbf{b}} +\lambda\, < \boldsymbol{\sigma}_1^h - \boldsymbol{\sigma}_2^h, \boldsymbol{\tau}_1^h >_{h,\mathbf{b},\Gamma_{12}-} = 0 \quad \forall \boldsymbol{\tau}_1^h \in \boldsymbol{\Sigma}_1^h\,, \end{aligned} \tag{3.25}$$

$$\begin{aligned} &(\boldsymbol{\sigma}_1^h, \mathbf{D}(\mathbf{v}_1^h)) + 2(1-\alpha)(\mathbf{D}(\mathbf{u}_1^h), \mathbf{D}(\mathbf{v}_1^h)) - (p_1^h, \nabla\cdot\mathbf{v}_1^h) + \frac{1}{\epsilon}(\mathbf{u}_1^h - \mathbf{u}_2^h, \mathbf{v}_1^h)_{\Gamma_0} \\ &\qquad = (\mathbf{f}, \mathbf{v}_1^h) \quad \forall \mathbf{v}_1^h \in \mathbf{X}_1^h\,, \end{aligned} \tag{3.26}$$

$$(q_1^h, \nabla\cdot\mathbf{u}_1^h) = 0 \quad \forall q_1 \in S_1^h(\Omega)\,, \tag{3.27}$$

and

$$\begin{aligned} &(\boldsymbol{\sigma}_2^h, \boldsymbol{\tau}_2^h) + \lambda\,((\mathbf{b}\cdot\nabla)\boldsymbol{\sigma}_2^h, \boldsymbol{\tau}_2^h)_h + \lambda\,(g_a(\boldsymbol{\sigma}_2^h, \nabla\mathbf{b}), \boldsymbol{\tau}_2^h) - 2\alpha(\mathbf{D}(\mathbf{u}_2^h), \boldsymbol{\tau}_2^h) \\ &\quad +\lambda\, < {\boldsymbol{\sigma}_2^h}^+ - {\boldsymbol{\sigma}_2^h}^-, {\boldsymbol{\tau}_2^h}^+ >_{h,\mathbf{b}} +\lambda\, < \boldsymbol{\sigma}_2^h - \boldsymbol{\sigma}_1^h, \boldsymbol{\tau}_2^h >_{h,\mathbf{b},\Gamma_{21}^-} = 0 \quad \forall \boldsymbol{\tau}_2^h \in \boldsymbol{\Sigma}_2^h\,, \end{aligned} \tag{3.28}$$

$$\begin{aligned} &(\boldsymbol{\sigma}_2^h, \mathbf{D}(\mathbf{v}_2^h)) + 2(1-\alpha)(\mathbf{D}(\mathbf{u}_2^h), \mathbf{D}(\mathbf{v}_2^h)) - (p_2^h, \nabla\cdot\mathbf{v}_2^h) - \frac{1}{\epsilon}(\mathbf{u}_1^h - \mathbf{u}_2^h, \mathbf{v}_2^h)_{\Gamma_0} \\ &\qquad = (\mathbf{f}, \mathbf{v}_2^h) \quad \forall \mathbf{v}_2^h \in \mathbf{X}_2^h\,, \end{aligned} \tag{3.29}$$

$$(q_2^h, \nabla\cdot\mathbf{u}_2^h) = 0 \quad \forall q_2 \in S_2^h(\Omega)\,. \tag{3.30}$$

Note that the stress boundary conditions (3.16), (3.20) are imposed weakly [11].

The next theorem shows the existence and uniqueness of a solution to the above coupled system (3.25)-(3.30). In proving the theorem we use the bilinear forms A_i defined on $\boldsymbol{\Sigma}_i^h \times \boldsymbol{V}_i^h$ for $i = 1,2$ by

$$\begin{aligned} A_i((\boldsymbol{\sigma}_i^h, \mathbf{u}_i^h), (\boldsymbol{\tau}_i^h, \mathbf{v}_i^h)) \;:=\; & (\boldsymbol{\sigma}_i^h, \boldsymbol{\tau}_i^h) - 2\alpha(\mathbf{D}(\mathbf{u}_i^h), \boldsymbol{\tau}_i^h) + 2\alpha(\boldsymbol{\sigma}_i^h, \mathbf{D}(\mathbf{v}_i^h)) \\ & +4\alpha(1-\alpha)\,(\mathbf{D}(\mathbf{u}_i^h), \mathbf{D}(\mathbf{v}_i^h))\; + \lambda\,(g_a(\boldsymbol{\sigma}_i^h, \nabla\mathbf{b}), \boldsymbol{\tau}_i^h). \end{aligned} \tag{3.31}$$

Note that A_i is continuous and coercive, if $1 - 2\lambda M d > 0$: since

$$(g_a(\boldsymbol{\sigma}_i^h, \nabla\mathbf{b}), \boldsymbol{\tau}_i^h) \le 2d\|\nabla\mathbf{b}\|_\infty \|\boldsymbol{\sigma}^h\|_0 \|\boldsymbol{\tau}^h\|_0 \le 2dM\|\boldsymbol{\sigma}^h\|_0\|\boldsymbol{\tau}^h\|_0\,, \tag{3.32}$$

we have

$$\begin{aligned} A_i((\boldsymbol{\sigma}_i^h, \mathbf{u}_i^h), (\boldsymbol{\tau}_i^h, \mathbf{v}_i^h)) \;\le\; & \|\boldsymbol{\sigma}_i^h\|_0\|\boldsymbol{\tau}_i^h\|_0 + 2\alpha\|\mathbf{D}(\mathbf{u}_i^h)\|_0\|\boldsymbol{\tau}_i^h\|_0 \\ & +2\alpha\|\boldsymbol{\sigma}_i^h\|_0\|\mathbf{D}(\mathbf{v}_i^h)\|_0 + 4\alpha(1-\alpha)\|\mathbf{D}(\mathbf{u}_i^h)\|_0\|\mathbf{D}(\mathbf{v}_i^h)\|_0 \\ & +2Md\lambda\|\boldsymbol{\sigma}_i^h\|_0\|\boldsymbol{\tau}_i^h\|_0 \\ \le\; & C(\|\boldsymbol{\sigma}_i^h\|_0 + \|\mathbf{u}\|_1)((\|\boldsymbol{\tau}_i^h\|_0 + \|\mathbf{v}\|_1) \\ \le\; & C\|(\boldsymbol{\sigma}_i^h, \mathbf{u}_i^h)\|_{\boldsymbol{\Sigma}_i\times\mathbf{X}_i^h}\|(\boldsymbol{\tau}_i^h, \mathbf{v}_i^h)\|_{\boldsymbol{\Sigma}_i\times\mathbf{X}_i}\,, \end{aligned} \tag{3.33}$$

where $\|(\boldsymbol{\tau}_i^h, \mathbf{v}_i^h)\|_{\boldsymbol{\Sigma}_i\times\mathbf{X}_i}$ is defined as $(\|\boldsymbol{\tau}_i^h\|_0^2 + \|\mathbf{v}_i^h\|_1^2)^{1/2}$. Also, using (3.32),

$$\begin{aligned} A_i((\boldsymbol{\sigma}_i^h, \mathbf{u}_i^h), (\boldsymbol{\sigma}_i^h, \mathbf{u}_i^h)) &= \|\boldsymbol{\sigma}_i^h\|_0^2 + \lambda(g_a(\boldsymbol{\sigma}_i^h, \nabla\mathbf{b}), \boldsymbol{\sigma}_i^h) + 4\alpha(1-\alpha)\|\mathbf{D}(\mathbf{u}_i^h)\|_0^2 \\ &\geq \|\boldsymbol{\sigma}_i^h\|_0^2 - 2\lambda M d\|\boldsymbol{\sigma}_i^h\|_0^2 + 4\alpha(1-\alpha)\|\mathbf{D}(\mathbf{u}_i^h)\|_0^2 \\ &\geq C\|(\boldsymbol{\sigma}_i^h, \mathbf{u}_i^h)\|^2_{\boldsymbol{\Sigma}_i\times\mathbf{X}_i}, \end{aligned} \tag{3.34}$$

if $1 - 2\lambda M d > 0$. In the proof of the next theorem we will also use the inverse estimate [3], [9]

$$\|\nabla\boldsymbol{\sigma}_i^h\|_{0,h} \leq Ch^{-1}\|\boldsymbol{\sigma}_i^h\|_0 \quad \text{for } \boldsymbol{\sigma}_i^h \in \boldsymbol{\Sigma}_i^h, \tag{3.35}$$

the local inverse inequality [21],

$$\|\boldsymbol{\sigma}_i^h\|_{0,\partial K} \;\leq\; Ch^{-1/2}\|\boldsymbol{\sigma}_i^h\|_{0,K} \quad \text{for } \boldsymbol{\sigma}_i^h \in \boldsymbol{\Sigma}_i^h, \tag{3.36}$$

and the trace theorem [9],

$$\|\mathbf{u}_i\|_{0,\Gamma_i\cup\Gamma_0} \;\leq\; C\|\mathbf{u}_i\|_1 \quad \text{for } \mathbf{u}_i \in \mathbf{X}_i\,. \tag{3.37}$$

Theorem 3.1. *The system (3.25)-(3.30) has a unique solution, if* $1 - 2\lambda M d > 0$.

Proof. Using the *div free* spaces $\boldsymbol{V}_i^h$ $i = 1, 2$, the coupled system (3.25)-(3.30) is equivalent to

$$\begin{aligned} &\sum_{i=1}^{2}\Big[A_i((\boldsymbol{\sigma}_i^h, \mathbf{u}_i^h), (\boldsymbol{\tau}_i^h, \mathbf{v}_i^h)) + \lambda((\mathbf{b}\cdot\nabla)\boldsymbol{\sigma}_i^h, \boldsymbol{\tau}_i^h)_h + \lambda < {\boldsymbol{\sigma}_i^h}^+ - {\boldsymbol{\sigma}_i^h}^-, {\boldsymbol{\tau}_i^h}^+ >_{h,\mathbf{b}}\Big] \\ &\quad +\lambda\left[< \boldsymbol{\sigma}_1^h - \boldsymbol{\sigma}_2^h, \boldsymbol{\tau}_1^h >_{h,\mathbf{b},\Gamma_{12}^-} + < \boldsymbol{\sigma}_2^h - \boldsymbol{\sigma}_1^h, \boldsymbol{\tau}_2^h >_{h,\mathbf{b},\Gamma_{21}^-}\right] + \frac{1}{\epsilon}(\mathbf{u}_1^h - \mathbf{u}_2^h, \mathbf{v}_1^h - \mathbf{v}_2^h)_{\Gamma_0} \\ &\qquad = \sum_{i=1}^{2} F_i((\boldsymbol{\tau}_i^h, \mathbf{v}_i^h)) \quad \forall(\boldsymbol{\tau}_i^h, \mathbf{v}_i^h) \in \boldsymbol{\Sigma}_i^h \times \boldsymbol{V}_i^h, \end{aligned} \tag{3.38}$$

where $F_i(\cdot) : \boldsymbol{\Sigma}_i^h \times \boldsymbol{V}_i^h \to R$ is a functional defined by

$$F_i((\boldsymbol{\tau}_i^h, \mathbf{v}_i^h)) := 2\alpha(\mathbf{f}, \mathbf{v}_i^h).$$

It is straight forward to show F_i is bounded:

$$|F_i((\boldsymbol{\tau}_i^h, \mathbf{v}_i^h))| \leq 2\alpha\|\mathbf{f}\|_{-1}\|\mathbf{v}_i^h\|_1 \leq 2\alpha\|\mathbf{f}\|_{-1}\|(\boldsymbol{\tau}_i^h, \mathbf{v}_i^h)\|_{\boldsymbol{\Sigma}_i\times\mathbf{X}_i}. \tag{3.39}$$

We will show the bilinear form in (3.38) is continuous and coercive in $\prod_{i=1}^2(\boldsymbol{\Sigma}_i^h \times \boldsymbol{V}_i^h)$, if $1-2\lambda M d > 0$. Using (3.35) and (3.36),

$$\begin{aligned} &((\mathbf{b}\cdot\nabla)\boldsymbol{\sigma}_i^h, \boldsymbol{\tau}_i^h)_h + < {\boldsymbol{\sigma}_i^h}^+ - {\boldsymbol{\sigma}_i^h}^-, {\boldsymbol{\tau}_i^h}^+ >_{h,\mathbf{b}} \\ &\qquad \leq C\left[\|\mathbf{b}\|_\infty\|\nabla\boldsymbol{\sigma}_i^h\|_{0,h}\|\boldsymbol{\tau}_i^h\|_0 + \|\mathbf{b}\|_\infty\|\boldsymbol{\sigma}_i^h\|_{0,\Gamma_i^h}\|\boldsymbol{\tau}_i^h\|_{0,\Gamma_i^h}\right] \\ &\qquad \leq C\left[\|\mathbf{b}\|_\infty(h^{-1}\|\boldsymbol{\sigma}_i^h\|_0)\|\boldsymbol{\tau}_i^h\|_0 + \|\mathbf{b}\|_\infty(h^{-1/2}\|\boldsymbol{\sigma}_i^h\|_0)(h^{-1/2}\|\boldsymbol{\tau}_i^h\|_0)\right] \\ &\qquad \leq CMh^{-1}\|\boldsymbol{\sigma}_i^h\|_0\|\boldsymbol{\tau}_i^h\|_0, \end{aligned} \tag{3.40}$$

and

$$\begin{aligned} &< \boldsymbol{\sigma}_1^h - \boldsymbol{\sigma}_2^h, \boldsymbol{\tau}_1^h >_{h,\mathbf{b},\Gamma_{12}^-} + < \boldsymbol{\sigma}_2^h - \boldsymbol{\sigma}_1^h, \boldsymbol{\tau}_2^h >_{h,\mathbf{b},\Gamma_{21}^-} \\ &\leq C\left[\|\mathbf{b}\|_\infty(\|\boldsymbol{\sigma}_1^h\|_{0,\Gamma_0} + \|\boldsymbol{\sigma}_2^h\|_{0,\Gamma_0})\|\boldsymbol{\tau}_1^h\|_{0,\Gamma_0} + \|\mathbf{b}\|_\infty(\|\boldsymbol{\sigma}_2^h\|_{0,\Gamma_0} + \|\boldsymbol{\sigma}_1^h\|_{0,\Gamma_0})\|\boldsymbol{\tau}_2^h\|_{0,\Gamma_0}\right] \\ &\leq CM\Big[(h^{-1/2}\|\boldsymbol{\sigma}_1^h\|_0 + h^{-1/2}\|\boldsymbol{\sigma}_2^h\|_0)(h^{-1/2}\|\boldsymbol{\tau}_1^h\|_0) \\ &\qquad\qquad + (h^{-1/2}\|\boldsymbol{\sigma}_2^h\|_0 + h^{-1/2}\|\boldsymbol{\sigma}_1^h\|_0)(h^{-1/2}\|\boldsymbol{\tau}_2^h\|_0)\Big] \\ &\leq CMh^{-1}\sum_{i,j=1}^{2}\|\boldsymbol{\sigma}_i^h\|_0\|\boldsymbol{\tau}_j^h\|_0\,. \end{aligned} \tag{3.41}$$

Also, by (3.37),

$$(\mathbf{u}_1^h - \mathbf{u}_2^h, \mathbf{v}_1^h - \mathbf{v}_2^h)_{\Gamma_0} \le C \sum_{i,j=1}^{2} \|\mathbf{u}_i^h\|_1 \|\mathbf{v}_j^h\|_1 \,. \tag{3.42}$$

Hence, by (3.33) and (3.40)-(3.42),

$$\begin{aligned}
&\sum_{i=1}^{2} \Big[A_i((\boldsymbol{\sigma}_i^h, \mathbf{u}_i^h), (\boldsymbol{\tau}_i^h, \mathbf{v}_i^h)) + \lambda((\mathbf{b}\cdot\nabla)\boldsymbol{\sigma}_i^h, \boldsymbol{\tau}_i^h)_h + \lambda < \boldsymbol{\sigma}_i^{h^+} - \boldsymbol{\sigma}_i^{h^-}, \boldsymbol{\tau}_i^{h^+} >_{h,\mathbf{b}} \Big] \\
&\quad + \lambda \Big[< \boldsymbol{\sigma}_1^h - \boldsymbol{\sigma}_2^h, \boldsymbol{\tau}_1^h >_{h,\mathbf{b},\Gamma_{12}^-} + < \boldsymbol{\sigma}_2^h - \boldsymbol{\sigma}_1^h, \boldsymbol{\tau}_2^h >_{h,\mathbf{b},\Gamma_{21}^-} \Big] + \frac{1}{\epsilon}(\mathbf{u}_1^h - \mathbf{u}_2^h, \mathbf{v}_1^h - \mathbf{v}_2^h)_{\Gamma_0} \\
\le\ & C \left[\sum_{i=1}^{2} (\|\boldsymbol{\sigma}_i^h\|_0 + \|\mathbf{u}_i^h\|_1)(\|\boldsymbol{\tau}_i^h\|_0 + \|\mathbf{v}_i^h\|_1) + Mh^{-1} \sum_{i,j=1}^{2} \|\boldsymbol{\sigma}_i^h\|_0 \|\boldsymbol{\tau}_i^h\|_0 \right] \\
\le\ & C^h \sum_{i,j=1}^{2} (\|\boldsymbol{\sigma}_i^h\|_0 + \|\mathbf{u}_i^h\|_1)(\|\boldsymbol{\tau}_i^h\|_0 + \|\mathbf{v}_i^h\|_1) \\
\le\ & C^h \|(\boldsymbol{\sigma}_1^h, \mathbf{u}_1^h, \boldsymbol{\sigma}_2^h, \mathbf{u}_2^h)\|_{\boldsymbol{\Sigma}_1\times\mathbf{X}_1\times\boldsymbol{\Sigma}_2\times\mathbf{X}_2} \|(\boldsymbol{\tau}_1^h, \mathbf{v}_1^h, \boldsymbol{\tau}_2^h, \mathbf{v}_2^h)\|_{\boldsymbol{\Sigma}_1\times\mathbf{X}_1\times\boldsymbol{\Sigma}_2\times\mathbf{X}_2} \,,
\end{aligned} \tag{3.43}$$

where C^h is a constant dependent on h.

We now show the coercivity of the bilinear form. First note, using integration by parts, that

$$\begin{aligned}
&\sum_{i=1}^{2} \Big[((\mathbf{b}\cdot\nabla)\boldsymbol{\sigma}_i^h, \boldsymbol{\tau}_i^h)_h + < \boldsymbol{\sigma}_i^{h^+} - \boldsymbol{\sigma}_i^{h^-}, \boldsymbol{\tau}_i^{h^+} >_{h,\mathbf{b}} \Big] \\
&\qquad\qquad + < \boldsymbol{\sigma}_1^h - \boldsymbol{\sigma}_2^h, \boldsymbol{\tau}_1^h >_{h,\mathbf{b},\Gamma_{12}^-} + < \boldsymbol{\sigma}_2^h - \boldsymbol{\sigma}_1^h, \boldsymbol{\tau}_2^h >_{h,\mathbf{b},\Gamma_{21}^-} \\
&= \sum_{i=1}^{2} \Big[-((\mathbf{b}\cdot\nabla)\boldsymbol{\tau}_i^h, \boldsymbol{\sigma}_i^h)_h + < \boldsymbol{\sigma}_i^{h^-}, \boldsymbol{\tau}_i^{h^-} - \boldsymbol{\tau}_i^{h^+} >_{h,\mathbf{b}} \Big] \\
&\qquad\qquad + < \boldsymbol{\sigma}_2^h, \boldsymbol{\tau}_2^h - \boldsymbol{\tau}_1^h >_{h,\mathbf{b},\Gamma_{12}^-} + < \boldsymbol{\sigma}_1^h, \boldsymbol{\tau}_1^h - \boldsymbol{\tau}_2^h >_{h,\mathbf{b},\Gamma_{21}^-} \ .
\end{aligned} \tag{3.44}$$

Hence, if $\boldsymbol{\tau}_i^h = \boldsymbol{\sigma}_i^h$, we obtain

$$\begin{aligned}
&\sum_{i=1}^{2} \Big[((\mathbf{b}\cdot\nabla)\boldsymbol{\sigma}_i^h, \boldsymbol{\sigma}_i^h)_h + < \boldsymbol{\sigma}_i^{h^+} - \boldsymbol{\sigma}_i^{h^-}, \boldsymbol{\sigma}_i^{h^+} >_{h,\mathbf{b}} \Big] \\
&\qquad\qquad + < \boldsymbol{\sigma}_1^h - \boldsymbol{\sigma}_2^h, \boldsymbol{\sigma}_1^h >_{h,\mathbf{b},\Gamma_{12}^-} + < \boldsymbol{\sigma}_2^h - \boldsymbol{\sigma}_1^h, \boldsymbol{\sigma}_2^h >_{h,\mathbf{b},\Gamma_{21}^-} \\
&= \frac{1}{2} \left[\sum_{i=1}^{2} \ll \boldsymbol{\sigma}_i^{h^+} - \boldsymbol{\sigma}_i^{h^-} \gg_{h,\mathbf{b}}^2 + \ll \boldsymbol{\sigma}_1^h - \boldsymbol{\sigma}_2^h \gg_{h,\mathbf{b},\Gamma_{12}^-}^2 + \ll \boldsymbol{\sigma}_2^h - \boldsymbol{\sigma}_1^h \gg_{h,\mathbf{b},\Gamma_{21}^-}^2 \right] \\
&\ge\ 0 \,.
\end{aligned} \tag{3.45}$$

Now the coercivity of the bilinear form follows from (3.34) and (3.45):

$$\begin{aligned}
&\sum_{i=1}^{2}\Big[A_i((\boldsymbol{\sigma}_i^h,\mathbf{u}_i^h),(\boldsymbol{\sigma}_i^h,\mathbf{u}_i^h))+\lambda((\mathbf{b}\cdot\nabla)\boldsymbol{\sigma}_i^h,\boldsymbol{\sigma}_i^h)+\lambda <\boldsymbol{\sigma}_i^{h+}-\boldsymbol{\sigma}_i^{h-},\boldsymbol{\sigma}_i^{h+}>_{h,\mathbf{b}}\Big]\\
&\quad+\lambda\Big[<\boldsymbol{\sigma}_1^h-\boldsymbol{\sigma}_2^h,\boldsymbol{\sigma}_1^h>_{h,\mathbf{b},\Gamma_{12}^-}+<\boldsymbol{\sigma}_2^h-\boldsymbol{\sigma}_1^h,\boldsymbol{\sigma}_2^h>_{h,\mathbf{b},\Gamma_{21}^-}\Big]+\frac{1}{\epsilon}(\mathbf{u}_1^h-\mathbf{u}_2^h,\mathbf{u}_1^h-\mathbf{u}_2^h)_{\Gamma_0}\\
&\geq\sum_{i=1}^{2}\Big[\|\boldsymbol{\sigma}_i^h\|_0^2-2\lambda Md\|\boldsymbol{\sigma}_i^h\|_0^2+4\alpha(1-\alpha)\|\mathbf{D}(\mathbf{u}_i^h)\|_0^2+\frac{\lambda}{2}\ll\boldsymbol{\sigma}_i^{h+}-\boldsymbol{\sigma}_i^{h-}\gg_{h,\mathbf{b}}^2\Big]\\
&\quad+\frac{\lambda}{2}\Big[\ll\boldsymbol{\sigma}_1^h-\boldsymbol{\sigma}_2^h\gg_{h,\mathbf{b},\Gamma_{12}^-}^2+\ll\boldsymbol{\sigma}_2^h-\boldsymbol{\sigma}_1^h\gg_{h,\mathbf{b},\Gamma_{21}^-}^2\Big]+\frac{1}{\epsilon}(\mathbf{u}_1^h-\mathbf{u}_2^h)_{\Gamma_0}^2\\
&\geq\sum_{i=1}^{2}\big[(1-2\lambda Md)\|\boldsymbol{\sigma}_i^h\|_0^2+4\alpha(1-\alpha)\|\mathbf{D}(\mathbf{u}_i^h)\|_0^2\big]\\
&\geq C\sum_{i=1}^{2}\|(\boldsymbol{\sigma}_i^h,\mathbf{u}_i^h)\|_{\boldsymbol{\Sigma}_i\times\mathbf{X}_i}^2\\
&\geq C\,\|(\boldsymbol{\sigma}_1^h,\mathbf{u}_1^h,\boldsymbol{\sigma}_2^h,\mathbf{u}_2^h)\|_{\boldsymbol{\Sigma}_1\times\mathbf{X}_1\times\boldsymbol{\Sigma}_2\times\mathbf{X}_2}^2\,. \qquad (3.46)
\end{aligned}$$

Therefore, by the Lax-Milgram theorem, there exists a unique $(\boldsymbol{\sigma}_i^h,\mathbf{u}_i^h)\in\boldsymbol{\Sigma}_i^h\times\boldsymbol{V}_i^h$, if $1-2\lambda Md>0$. Finally, existence of a unique solution $(\boldsymbol{\sigma}_i^h,\mathbf{u}_i^h,p_i^h)\in\boldsymbol{\Sigma}_i^h\times\mathbf{X}_i^h\times S_i^h$ follows from the inf-sup condition (3.24). □

4 Convergence Analysis

We will prove the solution of the two-domain problem (3.25)-(3.30) converges to the solution $(\boldsymbol{\sigma}^{ex},\mathbf{u}^{ex},p^{ex})$ satisfying (2.5)-(2.8). Let $\boldsymbol{\sigma}_i^{ex}=\boldsymbol{\sigma}^{ex}|_{\Omega_i\cup\Gamma_0}$, $\mathbf{u}_i^{ex}=\mathbf{u}^{ex}|_{\Omega_i\cup\Gamma_0}$ and $p_i^{ex}=p^{ex}|_{\Omega_i\cup\Gamma_0}$ for $i=1,2$. Define

$$\mathbf{g}^{ex}:=(\boldsymbol{\sigma}^{ex}+2(1-\alpha)\mathbf{D}(\mathbf{u}^{ex})+p^{ex}\mathbf{I})\cdot\mathbf{n}_1 \qquad \text{on } \Gamma_0.$$

Note that, for $i=1,2$, $(\boldsymbol{\sigma}_i^{ex},\mathbf{u}_i^{ex},p_i^{ex})$ satisfy

$$(\boldsymbol{\sigma}_i^{ex},\boldsymbol{\tau}_i)+\lambda((\mathbf{b}\cdot\nabla)\boldsymbol{\sigma}_i^{ex},\boldsymbol{\tau}_i)+\lambda(g_a(\boldsymbol{\sigma}_i^{ex},\nabla\mathbf{b}),\boldsymbol{\tau}_i)-2\alpha(\mathbf{D}(\mathbf{u}_i^{ex}),\boldsymbol{\tau}_i)=0 \quad \forall\boldsymbol{\tau}_i\in\boldsymbol{\Sigma}_i\,, \qquad (4.1)$$

$$\begin{aligned}(\boldsymbol{\sigma}_i^{ex},\mathbf{D}(\mathbf{v}_i))+2(1-\alpha)(\mathbf{D}(\mathbf{u}_i^{ex}),\mathbf{D}(\mathbf{v}_i))-(p_i^{ex},\nabla\cdot\mathbf{v}_i)&\\
=(\mathbf{f},\mathbf{v}_i)+(-1)^{i+1}(\mathbf{g}^{ex},\mathbf{v}_i)_{\Gamma_0} \quad \forall\mathbf{v}_i\in\mathbf{X}_i\,,& \qquad (4.2)\end{aligned}$$

$$(q_i,\nabla\cdot\mathbf{u}_i^{ex})=0 \quad \forall q_i\in S_i\,. \qquad (4.3)$$

In the *div free* spaces $\boldsymbol{V}_i$, (4.1)-(4.3) is equivalent to

$$\sum_{i=1}^{2}[A_i((\boldsymbol{\sigma}_i^{ex},\mathbf{u}_i^{ex}),(\boldsymbol{\tau}_i,\mathbf{v}_i))+\lambda((\mathbf{b}\cdot\nabla)\boldsymbol{\sigma}_i^{ex},\boldsymbol{\tau}_i)]=2\alpha(\mathbf{f},\mathbf{v}_i)+2\alpha(\mathbf{g}^{ex},\mathbf{v}_1-\mathbf{v}_2)_{\Gamma_0}. \qquad (4.4)$$

We introduce some approximation properties (see [3] or [9]), which will be used in order to prove the next theorem. If $\tilde{\boldsymbol{\sigma}}_i^h\in\boldsymbol{\Sigma}_i^h$ is the orthogonal projection of $\boldsymbol{\sigma}_i^{ex}$ on T_i^h in $\boldsymbol{\Sigma}_i$, and $\tilde{\mathbf{u}}_i^h\in\boldsymbol{V}_i^h$ is defined as the interpolant of $\mathbf{u}_i^{ex}$ in $\boldsymbol{V}_i$, then we have the following standard results. For $\mathbf{u}^{ex}\in\mathbf{H}^3(\Omega_i)$ and $\boldsymbol{\sigma}^{ex}\in\mathbf{H}^2(\Omega_i)$

$$\begin{aligned}\|\nabla(\mathbf{u}_i^{ex}-\tilde{\mathbf{u}}_i^h)\|_0 &\leq C\,h^2\|\mathbf{u}_i^{ex}\|_3\,, & (4.5)\\
\|\boldsymbol{\sigma}_i^{ex}-\tilde{\boldsymbol{\sigma}}_i^h\|_0+h\|\nabla(\boldsymbol{\sigma}_i^{ex}-\tilde{\boldsymbol{\sigma}}_i^h)\|_0 &\leq Ch^2\|\boldsymbol{\sigma}_i^{ex}\|_2\,. & (4.6)\end{aligned}$$

Theorem 4.1. *If* $1-2\lambda Md > 0$ *and* $(\boldsymbol{\sigma}_i^h, \mathbf{u}_i^h)$ *for* $i = 1, 2$*, satisfy the coupled decomposition problem (3.25)-(3.30), they converge to the exact solution* $(\boldsymbol{\sigma}^{ex}, \mathbf{u}^{ex})$ *as* ϵ *goes to* 0*. Furthermore, we have the estimate*

$$\sum_{i=1}^{2} \left[\|\boldsymbol{\sigma}_i^{ex} - \boldsymbol{\sigma}_i^h\|_0 + \|\mathbf{u}_i^{ex} - \mathbf{u}_i^h\|_1 \right] \le C(h + \sqrt{\epsilon}\|\mathbf{g}^{ex}\|_{\Gamma_0}). \tag{4.7}$$

Proof. Subtracting (3.38) from (4.4), and adding and subtracting $\widetilde{\boldsymbol{\sigma}}_i^h$, $\widetilde{\mathbf{u}}_i^h$, imply

$$\begin{aligned}
&\sum_{i=1}^{2} \Big[A_i((\widetilde{\boldsymbol{\sigma}}_i^h - \boldsymbol{\sigma}_i^h, \widetilde{\mathbf{u}}_i^h - \mathbf{u}_i^h), (\boldsymbol{\tau}_i^h, \mathbf{v}_i^h)) \\
&\qquad + \lambda((\mathbf{b}\cdot\nabla)(\widetilde{\boldsymbol{\sigma}}_i^h - \boldsymbol{\sigma}_i^h), \boldsymbol{\tau}_i^h) + \lambda < (\widetilde{\boldsymbol{\sigma}}_i^h - \boldsymbol{\sigma}_i^h)^+ - (\widetilde{\boldsymbol{\sigma}}_i^h - \boldsymbol{\sigma}_i^h)^-, {\boldsymbol{\tau}_i^h}^+ >_{h,\mathbf{b}} \Big] \\
&\quad + \lambda \Big[< (\widetilde{\boldsymbol{\sigma}}_1^h - \boldsymbol{\sigma}_1^h) - (\widetilde{\boldsymbol{\sigma}}_2^h - \boldsymbol{\sigma}_2^h), \boldsymbol{\tau}_1^h >_{h,\mathbf{b},\Gamma_{12}^-} + < (\widetilde{\boldsymbol{\sigma}}_2^h - \boldsymbol{\sigma}_2^h) - (\widetilde{\boldsymbol{\sigma}}_1^h - \boldsymbol{\sigma}_1^h), \boldsymbol{\tau}_2^h >_{h,\mathbf{b},\Gamma_{21}^-} \Big] \\
&\quad - \frac{2\alpha}{\epsilon}(\mathbf{u}_1^h - \mathbf{u}_2^h, \mathbf{v}_1^h - \mathbf{v}_2^h)_{\Gamma_0} \\
&= \sum_{i=1}^{2} \Big[A_i((\widetilde{\boldsymbol{\sigma}}_i^h - \boldsymbol{\sigma}_i^{ex}, \widetilde{\mathbf{u}}_i^h - \mathbf{u}_i^{ex}), (\boldsymbol{\tau}_i^h, \mathbf{v}_i^h)) \\
&\qquad + \lambda((\mathbf{b}\cdot\nabla)(\widetilde{\boldsymbol{\sigma}}_i^h - \boldsymbol{\sigma}_i^{ex}), \boldsymbol{\tau}_i^h) + \lambda < (\widetilde{\boldsymbol{\sigma}}_i^h - \boldsymbol{\sigma}_i^{ex})^+ - (\widetilde{\boldsymbol{\sigma}}_i^h - \boldsymbol{\sigma}_i^{ex})^-, {\boldsymbol{\tau}_i^h}^+ >_{h,\mathbf{b}} \Big] \\
&\quad + \lambda \Big[< (\widetilde{\boldsymbol{\sigma}}_1^h - \boldsymbol{\sigma}_1^{ex}) - (\widetilde{\boldsymbol{\sigma}}_2^h - \boldsymbol{\sigma}_2^{ex}), \boldsymbol{\tau}_1^h >_{h,\mathbf{b},\Gamma_{12}^-} + < (\widetilde{\boldsymbol{\sigma}}_2^h - \boldsymbol{\sigma}_2^{ex}) - (\widetilde{\boldsymbol{\sigma}}_1^h - \boldsymbol{\sigma}_1^{ex}), \boldsymbol{\tau}_2^h >_{h,\mathbf{b},\Gamma_{21}^-} \Big] \\
&\qquad + 2\alpha(\mathbf{g}^{ex}, \mathbf{v}_1^h - \mathbf{v}_2^h)_{\Gamma_0} \qquad \forall(\boldsymbol{\tau}_i^h, \mathbf{v}_i^h) \in \boldsymbol{\Sigma}_i^h \times \boldsymbol{V}_i^h\,,
\end{aligned} \tag{4.8}$$

where we used $\boldsymbol{\sigma}_1^{ex} = \boldsymbol{\sigma}_2^{ex}$ on Γ_0 and that $\boldsymbol{\sigma}_i^{ex}$ is a continuous function. Let $\boldsymbol{\tau}_i^h = \widetilde{\boldsymbol{\sigma}}_i^h - \boldsymbol{\sigma}_i^h$, $\mathbf{v}_i^h = \widetilde{\mathbf{u}}_i^h - \mathbf{u}_i^h$. We have a lower bound of the left hand side of (4.8) using (3.34) and (3.45):

$$\text{LHS} \ge \sum_{i=1}^{2} \Big[(1-2\lambda Md)\|\widetilde{\boldsymbol{\sigma}}_i^h - \boldsymbol{\sigma}_i^h\|_0^2 + 4\alpha(1-\alpha)\|\mathbf{D}(\widetilde{\mathbf{u}}_i^h - \mathbf{u}_i^h)\|_0^2 \Big] + \frac{2\alpha}{\epsilon}\|\mathbf{u}_1^h - \mathbf{u}_2^h\|_{\Gamma_0}^2\,. \tag{4.9}$$

Estimates for the jump terms in the right side of (4.8) are obtained in the similar way shown in (3.40) and (3.41):

$$\begin{aligned}
< (\widetilde{\boldsymbol{\sigma}}_i^h - \boldsymbol{\sigma}_i^{ex})^+ - (\widetilde{\boldsymbol{\sigma}}_i^h - \boldsymbol{\sigma}_i^{ex})^-, \widetilde{\boldsymbol{\sigma}}_i^h - {\boldsymbol{\sigma}_i^h}^+ >_{h,\mathbf{b}} &\le C\|\mathbf{b}\|_\infty \|\widetilde{\boldsymbol{\sigma}}_i^h - \boldsymbol{\sigma}_i^{ex}\|_{0,\Gamma_i^h} \|\widetilde{\boldsymbol{\sigma}}_i^h - \boldsymbol{\sigma}_i^h\|_{0,\Gamma_i^h} \\
&\le CMh^{-1}\|\widetilde{\boldsymbol{\sigma}}_i^h - \boldsymbol{\sigma}_i^{ex}\|_0 \|\widetilde{\boldsymbol{\sigma}}_i^h - \boldsymbol{\sigma}_i^h\|_0,
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
&< (\widetilde{\boldsymbol{\sigma}}_1^h - \boldsymbol{\sigma}_1^{ex}) - (\widetilde{\boldsymbol{\sigma}}_2^h - \boldsymbol{\sigma}_2^{ex}), \widetilde{\boldsymbol{\sigma}}_1^h - \boldsymbol{\sigma}_1^h >_{h,\mathbf{b},\Gamma_{12}^-} \\
&\qquad\qquad + < (\widetilde{\boldsymbol{\sigma}}_2^h - \boldsymbol{\sigma}_2^{ex}) - (\widetilde{\boldsymbol{\sigma}}_1^h - \boldsymbol{\sigma}_1^{ex}), \widetilde{\boldsymbol{\sigma}}_2^h - \boldsymbol{\sigma}_2^h >_{h,\mathbf{b},\Gamma_{21}^-} \\
&\le C\|\mathbf{b}\|_\infty (\|\widetilde{\boldsymbol{\sigma}}_1^h - \boldsymbol{\sigma}_1^{ex}\|_{0,\Gamma_0} + \|\widetilde{\boldsymbol{\sigma}}_2^h - \boldsymbol{\sigma}_2^{ex}\|_{0,\Gamma_0})(\|\widetilde{\boldsymbol{\sigma}}_1^h - \boldsymbol{\sigma}_1^h\|_{0,\Gamma_0} + \|\widetilde{\boldsymbol{\sigma}}_2^h - \boldsymbol{\sigma}_2^h\|_{0,\Gamma_0}) \\
&\le CMh^{-1}(\|\widetilde{\boldsymbol{\sigma}}_1^h - \boldsymbol{\sigma}_1^{ex}\|_0 + \|\widetilde{\boldsymbol{\sigma}}_2^h - \boldsymbol{\sigma}_2^{ex}\|_0)(\|\widetilde{\boldsymbol{\sigma}}_1^h - \boldsymbol{\sigma}_1^h\|_0 + \|\widetilde{\boldsymbol{\sigma}}_2^h - \boldsymbol{\sigma}_2^h\|_0)\,.
\end{aligned} \tag{4.11}$$

Using (3.33), (4.9)-(4.11) and the Young's inequality, we have

$$\sum_{i=1}^{2}\Big[(1-2\lambda Md)\|\widetilde{\boldsymbol\sigma}_i^h-\boldsymbol\sigma_i^h\|_0^2+4\alpha(1-\alpha)\|\mathbf{D}(\widetilde{\mathbf{u}}_i^h-\mathbf{u}_i^h)\|_0^2\Big]+\frac{2\alpha}{\epsilon}\|\mathbf{u}_1^h-\mathbf{u}_2^h\|_{\Gamma_0}^2$$
$$\le C\sum_{i=1}^{2}\Big[(\|\widetilde{\boldsymbol\sigma}_i^h-\boldsymbol\sigma_i^{ex}\|_0+\|\mathbf{D}(\widetilde{\mathbf{u}}_i^h-\mathbf{u}_i^{ex})\|_0)(\|\widetilde{\boldsymbol\sigma}_i^h-\boldsymbol\sigma_i^h\|_0+\|\mathbf{D}(\widetilde{\mathbf{u}}_i^h-\mathbf{u}_i^h)\|_0)$$
$$+\lambda M\|\nabla(\widetilde{\boldsymbol\sigma}_i^h-\boldsymbol\sigma_i^{ex})\|_0\|\widetilde{\boldsymbol\sigma}_i^h-\boldsymbol\sigma_i^h\|_0+\lambda Mh^{-1}\|\widetilde{\boldsymbol\sigma}_i^h-\boldsymbol\sigma_i^{ex}\|_0\|\widetilde{\boldsymbol\sigma}_i^h-\boldsymbol\sigma_i^h\|_0\Big]$$
$$+C\lambda Mh^{-1}\sum_{i,j=1}^{2}\|\widetilde{\boldsymbol\sigma}_i^h-\boldsymbol\sigma_i^{ex}\|_0\|\widetilde{\boldsymbol\sigma}_j^h-\boldsymbol\sigma_j^h\|_0+2\alpha\|\mathbf{g}^{ex}\|_{\Gamma_0}\|\mathbf{u}_1-\mathbf{u}_2\|_{\Gamma_0}$$
$$\le C\sum_{i=1}^{2}\Big[\delta_1(\|\widetilde{\boldsymbol\sigma}_i^h-\boldsymbol\sigma_i^h\|_0+\|\mathbf{D}(\widetilde{\mathbf{u}}_i^h-\mathbf{u}_i^h)\|_0)^2+\frac{1}{4\delta_1}(\|\widetilde{\boldsymbol\sigma}_i^h-\boldsymbol\sigma_i^{ex}\|_0+\|\mathbf{D}(\widetilde{\mathbf{u}}_i^h-\mathbf{u}_i^{ex})\|_0)^2$$
$$+\delta_2\lambda M\|\widetilde{\boldsymbol\sigma}_i^h-\boldsymbol\sigma_i^h\|_0^2+\frac{\lambda M}{4\delta_2}\|\nabla(\widetilde{\boldsymbol\sigma}_i^h-\boldsymbol\sigma_i^{ex})\|_0^2+\delta_3\lambda M\|\widetilde{\boldsymbol\sigma}_i^h-\boldsymbol\sigma_i^h\|_0^2+\frac{\lambda Mh^{-2}}{4\delta_3}\|\widetilde{\boldsymbol\sigma}_i^h-\boldsymbol\sigma_i^{ex}\|_0^2\Big]$$
$$+C\sum_{i}^{2}\Big[\delta_4\lambda M\|\widetilde{\boldsymbol\sigma}_i^h-\boldsymbol\sigma_i^h\|_0^2+\frac{\lambda Mh^{-2}}{4\delta_4}\|\widetilde{\boldsymbol\sigma}_i^h-\boldsymbol\sigma_i^{ex}\|_0^2\Big]$$
$$+2\alpha\Big[\frac{1}{4\delta_5}\|\mathbf{u}_1^h-\mathbf{u}_2^h\|_{\Gamma_0}^2+\delta_5\|\mathbf{g}^{ex}\|_{\Gamma_0}^2\Big], \tag{4.12}$$

for arbitrary $\delta_i>0$, $i=1,2,\ldots,5$. If we take $\delta_5=\frac{\epsilon}{2}$,

$$\sum_{i=1}^{2}\Big[(1-2\lambda Md-2\delta_1C-(\delta_2+\delta_3+\delta_4)\lambda MC)\|\widetilde{\boldsymbol\sigma}_i^h-\boldsymbol\sigma_i^h\|_0^2$$
$$+(4\alpha(1-\alpha)-2C\delta_1)\|\mathbf{D}(\widetilde{\mathbf{u}}_i^h-\mathbf{u}_i^h)\|_0^2\Big]+\frac{\alpha}{\epsilon}\|\mathbf{u}_1^h-\mathbf{u}_2^h\|_{\Gamma_0}^2$$
$$\le C\sum_{i=1}^{2}\Big[\frac{1}{4\delta_1}(\|\widetilde{\boldsymbol\sigma}_i^h-\boldsymbol\sigma_i^{ex}\|_0+\|\mathbf{D}(\widetilde{\mathbf{u}}_i^h-\mathbf{u}_i^{ex})\|_0)^2+\frac{\lambda M}{4\delta_2}\|\nabla(\widetilde{\boldsymbol\sigma}_i^h-\boldsymbol\sigma_i^{ex})\|_0^2$$
$$+(\frac{\lambda Mh^{-2}}{4\delta_3}+\frac{\lambda Mh^{-2}}{4\delta_4})\|\widetilde{\boldsymbol\sigma}_i^h-\boldsymbol\sigma_i^{ex}\|_0^2\Big]+\alpha\epsilon\|\mathbf{g}^{ex}\|_{\Gamma_0}^2. \tag{4.13}$$

Therefore, using (4.5)-(4.6) and the triangle inequality, (4.7) follows. □

Remark 4.2. *The pressure on each subdomain, p_i^h, is determined up to a constant. In order to compute a pressure convergent to p^{ex} which has the zero mean value, construct $\overline{p}^h$ in$L_0^2(\Omega)$ by*

$$\overline{p}^h(\mathbf{x})=\begin{cases} p_1^h-K & \text{for } \mathbf{x}\in\overline{\Omega}_1\\ p_2^h-K & \text{for } \mathbf{x}\in\overline{\Omega}_2\end{cases},$$

where $K=\left(\int_{\Omega_1}p_1^h\,d\Omega_1+\int_{\Omega_2}p_2^h\,d\Omega_2\right)/meas(\Omega)$. Then, convergence of $\overline{p}^h$ to p^{ex} can be proved as shown in [17] or [18].

Remark 4.3. *The estimate (4.7) suggests the optimal scaling $\epsilon=Ch^2$ between ϵ and h.*

5 Iterative Algorithm

In this section we delete the superscript h in $(\boldsymbol\sigma_i^h,\mathbf{u}_i^h,p_i^h)$ to simplify our notations. Consider the following iterative algorithm.

Algorithm 5.1.

$$
\begin{aligned}
&(\boldsymbol{\sigma_1}^{n+1},\boldsymbol{\tau}^h)+\lambda((\mathbf{b}\cdot\nabla)\boldsymbol{\sigma}_1^{n+1},\boldsymbol{\tau}^h)+\lambda <\boldsymbol{\sigma}_1^{n+1+}-\boldsymbol{\sigma}_1^{n+1-},\boldsymbol{\tau}^{h+}>_{h,\mathbf{b}}\\
&\quad -2\alpha(\mathbf{D}(\mathbf{u_1}^{n+1}),\boldsymbol{\tau}^h)+\lambda<\boldsymbol{\sigma_1}^{n+1},\boldsymbol{\tau}^h>_{h,\mathbf{b},\Gamma_{12}^-}\\
&\qquad\qquad = -\lambda\,(g_a(\boldsymbol{\sigma_1}^n,\nabla\mathbf{b}),\boldsymbol{\tau}^h)+\lambda<\boldsymbol{\sigma_2}^n,\boldsymbol{\tau}^h>_{h,\mathbf{b},\Gamma_{12}^-}\quad \forall\boldsymbol{\tau}^h\in\boldsymbol{\Sigma}_1^h\,,
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
&(\boldsymbol{\sigma_1}^{n+1},\mathbf{D}(\mathbf{v}^h))+2(1-\alpha)(\mathbf{D}(\mathbf{u_1}^{n+1}),\mathbf{D}(\mathbf{v}^h))-(p_1{}^{n+1},\nabla\cdot\mathbf{v}^h)+\frac{1}{\epsilon}(\mathbf{u_1}^{n+1},\mathbf{v}^h)_{\Gamma_0}\\
&\qquad\qquad = (\mathbf{f},\mathbf{v}^h)+\frac{1}{\epsilon}(\mathbf{u_2}^n,\mathbf{v}^h)_{\Gamma_0}\quad\forall\mathbf{v}^h\in\mathbf{X}_1^h\,,
\end{aligned}
\tag{5.2}
$$

$$
(q^h,\nabla\cdot\mathbf{u_1}^{n+1})=0\quad\forall q\in S_1^h\,,
\tag{5.3}
$$

and

$$
\begin{aligned}
&(\boldsymbol{\sigma_2}^{n+1},\boldsymbol{\tau}^h)+\lambda((\mathbf{b}\cdot\nabla)\boldsymbol{\sigma}_2^{n+1},\boldsymbol{\tau}^h)+\lambda <\boldsymbol{\sigma}_2^{n+1+}-\boldsymbol{\sigma}_2^{n+1-},\boldsymbol{\tau}^{h+}>_{h,\mathbf{b}}\\
&\quad -2\alpha(\mathbf{D}(\mathbf{u_2}^{n+1}),\boldsymbol{\tau}^h)+\lambda<\boldsymbol{\sigma_2}^{n+1},\boldsymbol{\tau}^h>_{h,\mathbf{b},\Gamma_{21}^-}\\
&\qquad\qquad = -\lambda\,(g_a(\boldsymbol{\sigma_2}^n,\nabla\mathbf{b}),\boldsymbol{\tau}^h)+\lambda<\boldsymbol{\sigma_1}^n,\boldsymbol{\tau}^h>_{h,\mathbf{b},\Gamma_{21}^-}\quad \forall\boldsymbol{\tau}^h\in\boldsymbol{\Sigma}_2^h\,,
\end{aligned}
\tag{5.4}
$$

$$
\begin{aligned}
&(\boldsymbol{\sigma_2}^{n+1},\mathbf{D}(\mathbf{v}^h))+2(1-\alpha)(\mathbf{D}(\mathbf{u_2}^{n+1}),\mathbf{D}(\mathbf{v}^h))-(p_2{}^{n+1},\nabla\cdot\mathbf{v}^h)+\frac{1}{\epsilon}(\mathbf{u_2}^{n+1},\mathbf{v}^h)_{\Gamma_0}\\
&\qquad\qquad = (\mathbf{f},\mathbf{v}^h)+\frac{1}{\epsilon}(\mathbf{u_1}^n,\mathbf{v}^h)_{\Gamma_0}\quad\forall\mathbf{v}^h\in\mathbf{X}_2^h\,,
\end{aligned}
\tag{5.5}
$$

$$
(q^h,\nabla\cdot\mathbf{u}_2^{h\,n+1})=0\quad\forall q\in S_2^h\,.
\tag{5.6}
$$

In the proof of the next theorem we will use the Trace theorem

$$
\tilde{C}\|\mathbf{u}_i\|_{0,\Gamma_0}\le\|\mathbf{D}(\mathbf{u}_i)\|_0.
\tag{5.7}
$$

Theorem 5.2. *The iterative algorithm 5.1 converges for each fixed ϵ and h if λM is sufficiently small so that $\lambda Md+\frac{\lambda M}{2h}<1-\lambda Md$ and α satisfies $\lambda Md+\frac{\lambda M}{2h}-4\tilde{C}^2\alpha(1-\alpha)<\frac{\alpha}{\epsilon}<1-\lambda Md$, where $\tilde{C}$ is the constant in (5.7).*

Proof. In the discrete *div free* space V_i^h, subtracting (5.1)-(5.2) and (5.4)-(5.5) from (3.25)-(3.26) and (3.28)-(3.29) respectively, and letting $\boldsymbol{\tau}_i=\boldsymbol{\sigma}_i-\boldsymbol{\sigma_i}^{n+1}$, $\mathbf{v}_i=\mathbf{u}_i-\mathbf{u_i}^{n+1}$, we obtain

$$
\begin{aligned}
&\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma_1}^{n+1}\|_0^2+\frac{\lambda}{2}\ll(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_1^{n+1})^+-(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_1^{n+1})^-\gg_{h,\mathbf{b}}^2\\
&\quad -2\alpha(\mathbf{D}(\mathbf{u}_1-\mathbf{u_1}^{n+1}),\boldsymbol{\sigma}_1-\boldsymbol{\sigma_1}^{n+1})+\lambda\ll\boldsymbol{\sigma}_1-\boldsymbol{\sigma_1}^{n+1}\gg_{h,\mathbf{b},\Gamma_{12}^-}^2\\
&\quad = -\lambda\,(g_a(\boldsymbol{\sigma}_1-\boldsymbol{\sigma_1}^n,\nabla\mathbf{b}),\boldsymbol{\sigma}_1-\boldsymbol{\sigma_1}^{n+1})+\lambda<\boldsymbol{\sigma}_2-\boldsymbol{\sigma_2}^n,\boldsymbol{\sigma}_1-\boldsymbol{\sigma_1}^{n+1}>_{h,\mathbf{b},\Gamma_{12}^-}\,,
\end{aligned}
\tag{5.8}
$$

$$
\begin{aligned}
&(\boldsymbol{\sigma}_1-\boldsymbol{\sigma_1}^{n+1},\mathbf{D}(\mathbf{u}_1-\mathbf{u_1}^{n+1}))+2(1-\alpha)\|\mathbf{D}(\mathbf{u}_1-\mathbf{u_1}^{n+1})\|_0^2+\frac{1}{\epsilon}\|\mathbf{u}_1-\mathbf{u_1}^{n+1}\|_{\Gamma_0}^2\\
&\qquad = \frac{1}{\epsilon}(\mathbf{u}_2-\mathbf{u_2}^n,\mathbf{u}_1-\mathbf{u_1}^{n+1})_{\Gamma_0}\,,
\end{aligned}
\tag{5.9}
$$

and

$$\|\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_2{}^{n+1}\|_0^2 + \frac{\lambda}{2} \ll (\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_2^{n+1})^+ - (\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_2^{n+1})^- \gg_{h,\mathbf{b}}^2$$
$$-2\alpha(\mathbf{D}(\mathbf{u}_2 - \mathbf{u}_2{}^{n+1}), \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_2{}^{n+1}) + \lambda \ll \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_2{}^{n+1} \gg_{h,\mathbf{b},\Gamma_{21}^-}^2$$
$$= -\lambda\left(g_a(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_2{}^n, \nabla\mathbf{b}), \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_2{}^n\right) + \lambda < \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_1{}^n, \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_2{}^{n+1} >_{h,\mathbf{b},\Gamma_{21}^-}, \quad (5.10)$$

$$(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_2{}^{n+1}, \mathbf{D}(\mathbf{u}_2 - \mathbf{u}_2{}^{n+1})) + 2(1-\alpha)\|\mathbf{D}(\mathbf{u}_2 - \mathbf{u}_2{}^{n+1})\|_0^2 + \frac{1}{\epsilon}\|\mathbf{u}_2 - \mathbf{u}_2{}^{n+1}\|_{\Gamma_0}^2$$
$$= \frac{1}{\epsilon}(\mathbf{u}_1 - \mathbf{u}_1{}^n, \mathbf{u}_2 - \mathbf{u}_2{}^{n+1})_{\Gamma_0}. \quad (5.11)$$

Multiplying (5.9), (5.11) by 2α and adding all equations together, we have

$$\sum_{i=1}^{2}\Big[\|\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_i{}^{n+1}\|_0^2 + \frac{\lambda}{2} \ll (\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_i^{n+1})^+ - (\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_i^{n+1})^- \gg_{h,\mathbf{b}}^2$$
$$+4\alpha(1-\alpha)\|\mathbf{D}(\mathbf{u}_i - \mathbf{u}_i{}^{n+1})\|_0^2 + \frac{2\alpha}{\epsilon}\|\mathbf{u}_i - \mathbf{u}_i{}^{n+1}\|_{\Gamma_0}^2\Big]$$
$$+\lambda \ll \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_1{}^{n+1} \gg_{h,\mathbf{b},\Gamma_{12}^-}^2 + \lambda \ll \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_2{}^{n+1} \gg_{h,\mathbf{b},\Gamma_{21}^-}^2$$
$$\le \sum_{i=1}^{2} 2\lambda M d\|\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_i^n\|_0\|\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_i^{n+1}\|_0$$
$$+\lambda\left[< \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_2{}^n, \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_1{}^{n+1} >_{h,\mathbf{b},\Gamma_{12}^-} + < \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_1{}^n, \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_2{}^{n+1} >_{h,\mathbf{b},\Gamma_{21}^-}\right]$$
$$+\frac{2\alpha}{\epsilon}(\mathbf{u}_2 - \mathbf{u}_2{}^n, \mathbf{u}_1 - \mathbf{u}_1{}^{n+1})_{\Gamma_0} + \frac{2\alpha}{\epsilon}(\mathbf{u}_1 - \mathbf{u}_1{}^n, \mathbf{u}_2 - \mathbf{u}_2{}^{n+1})_{\Gamma_0}$$
$$\le \sum_{i=1}^{2} \lambda M d(\|\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_i^n\|_0^2 + \|\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_i^{n+1}\|_0^2)$$
$$+\frac{\lambda}{2}\Big[\ll \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_2{}^n \gg_{h,\mathbf{b},\Gamma_{12}^-}^2 + \ll \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_1{}^{n+1} \gg_{h,\mathbf{b},\Gamma_{12}^-}^2$$
$$+ \ll \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_1{}^n \gg_{h,\mathbf{b},\Gamma_{21}^-} + \ll \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_2{}^{n+1} \gg_{h,\mathbf{b},\Gamma_{21}^-}^2\Big]$$
$$+\frac{\alpha}{\epsilon}\left[\|\mathbf{u}_2 - \mathbf{u}_2{}^n\|_{\Gamma_0}^2 + \|\mathbf{u}_1 - \mathbf{u}_1{}^{n+1}\|_{\Gamma_0}^2 + \|\mathbf{u}_1 - \mathbf{u}_1{}^n\|_{\Gamma_0}^2 + \|\mathbf{u}_2 - \mathbf{u}_2{}^{n+1}\|_{\Gamma_0}^2\right], \quad (5.12)$$

by using (3.32), (3.34), (3.45), and the Young's inequality. Using (5.7), (5.12) becomes

$$\sum_{i=1}^{2}\left[(1-\lambda M d)\|\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_i{}^{n+1}\|_0^2 + \left(4\tilde{C}^2\alpha(1-\alpha) + \frac{\alpha}{\epsilon}\right)\|\mathbf{u}_i - \mathbf{u}_i{}^{n+1}\|_{\Gamma_0}^2\right]$$
$$+\frac{\lambda}{2}\left[\ll \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_1{}^{n+1} \gg_{h,\mathbf{b},\Gamma_{12}^-}^2 + \ll \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_2{}^{n+1} \gg_{h,\mathbf{b},\Gamma_{21}^-}^2\right]$$
$$\le \sum_{i=1}^{2} \lambda M d\,\|\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_i{}^n\|_0^2$$
$$+\frac{\lambda}{2}\left[\ll \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_2{}^n \gg_{h,\mathbf{b},\Gamma_{12}^-}^2 + \ll \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_1{}^n \gg_{h,\mathbf{b},\Gamma_{21}^-}^2\right] + \frac{\alpha}{\epsilon}\sum_{i=1}^{2}\|\mathbf{u}_i - \mathbf{u}_i{}^n\|_{\Gamma_0}^2.$$

As Γ_{12}^-, $\Gamma_{21}^- \subset \Gamma_0$, using the inverse estimate

$$\overline{C}h^{1/2}\|\boldsymbol{\sigma}_i^h\|_{0,\Gamma_0} \le \|\boldsymbol{\sigma}_i^h\|_{\Omega_i} \quad (5.13)$$

we obtain

$$\sum_{i=1}^{2}\left[(1-\lambda Md)\|\boldsymbol{\sigma}_i-\boldsymbol{\sigma}_i{}^{n+1}\|_0^2+\left(4\tilde{C}^2\alpha(1-\alpha)+\frac{\alpha}{\epsilon}\right)\|\mathbf{u}_i-\mathbf{u}_i{}^{n+1}\|_{\Gamma_0}^2\right]$$
$$\leq\sum_{1=2}^{2}\left[\left(\lambda Md+\frac{\lambda M}{2h}\right)\|\boldsymbol{\sigma}_i-\boldsymbol{\sigma}_i{}^{n}\|_0^2+\frac{\alpha}{\epsilon}\|\mathbf{u}_i-\mathbf{u}_i{}^{n}\|_{\Gamma_0}^2\right].$$

Note that

$$\frac{\frac{\alpha}{\epsilon}}{4\tilde{C}\alpha(1-\alpha)+\frac{\alpha}{\epsilon}}<1.$$

Hence $(\boldsymbol{\sigma}^{n+1},\mathbf{u}^{n+1})$ converges, if $\lambda Md+\frac{\lambda M}{2h}<1-\lambda Md$ and $\lambda Md+\frac{\lambda M}{2h}-4\tilde{C}^2\alpha(1-\alpha)<\frac{\alpha}{\epsilon}<1-\lambda Md$.

In order to prove convergence of pressure, subtract (5.2), (5.5) from (3.26), (3.29), respectively.

$$(\boldsymbol{\sigma}_i-\boldsymbol{\sigma}_i{}^{n+1},\mathbf{D}(\mathbf{v}_i))+2(1-\alpha)(\mathbf{D}(\mathbf{u}_i-\mathbf{u}_i{}^{n+1}),\mathbf{D}(\mathbf{v}_i))-(p_i-p_i^{n+1},\nabla\cdot\mathbf{v}_i)$$
$$+\frac{1}{\epsilon}(\mathbf{u}_i-\mathbf{u}_i{}^{n+1},\mathbf{v}_i)_{\Gamma_0}=\frac{1}{\epsilon}(\mathbf{u}_j-\mathbf{u}_j{}^{n},\mathbf{v}_i)_{\Gamma_0}, \tag{5.14}$$

where $(i,j)=(1,2)$ or $(2,1)$. Using the *inf-sup condition* (3.24), (5.7) and (5.14), we have

$$\begin{aligned}\|p_i-p_i^{n+1}\|_0 &\leq C\sup_{\mathbf{0}\neq\mathbf{v}_i^h\in\mathbf{X}_i^h}\frac{(p_i-p_i^{n+1},\nabla\cdot\mathbf{v}_i^h)}{\|\mathbf{v}_i^h\|_1}\\ &\leq C\sup_{\mathbf{0}\neq\mathbf{v}_i^h\in\mathbf{X}_i^h}\frac{1}{\|\mathbf{v}_i^h\|_1}\left[\|\boldsymbol{\sigma}_i-\boldsymbol{\sigma}_i{}^{n+1}\|_0+2(1-\alpha)\|\mathbf{D}(\mathbf{u}_i-\mathbf{u}_i{}^{n+1})\|_0\right.\\ &\quad\left.+\frac{1}{\tilde{C}\epsilon}\left(\|\mathbf{D}(\mathbf{u}_i-\mathbf{u}_i{}^{n+1})\|_0+\|\mathbf{D}(\mathbf{u}_j-\mathbf{u}_j{}^{n})\|_0\right)\right]\|\mathbf{D}(\mathbf{v}_i)\|.\end{aligned}$$

This implies

$$\sum_{i=1}^{2}\|p_i-p_i^{n+1}\|_0\leq C\sum_{i=1}^{2}\left[\|\boldsymbol{\sigma}_i-\boldsymbol{\sigma}_i{}^{n+1}\|_0+\|\mathbf{D}(\mathbf{u}_i-\mathbf{u}_i{}^{n+1})\|_0+\|\mathbf{D}(\mathbf{u}_i-\mathbf{u}_i{}^{n})\|_0\right].$$

Therefore, convergence of p^{n+1} follows from convergence of $(\boldsymbol{\sigma}^{n+1},\mathbf{u}^{n+1})$. □

As an alternate scheme, we may consider the slightly modified iterative algorithm.

Algorithm 5.3.

$$\begin{aligned}&(\boldsymbol{\sigma}_1{}^{n+1},\boldsymbol{\tau}^h)+\lambda((\mathbf{b}\cdot\nabla)\boldsymbol{\sigma}_1^{n+1},\boldsymbol{\tau}^h)+\lambda<\boldsymbol{\sigma}_1^{n+1^+}-\boldsymbol{\sigma}_1^{n+1^-},\boldsymbol{\tau}^{h^+}>_{h,\mathbf{b}}\\ &\quad+\lambda\left(g_a(\boldsymbol{\sigma}_1{}^{n+1},\nabla\mathbf{b}),\boldsymbol{\tau}^h\right)-2\alpha(\mathbf{D}(\mathbf{u}_1{}^{n+1}),\boldsymbol{\tau}^h)+\lambda<\boldsymbol{\sigma}_1{}^{n+1},\boldsymbol{\tau}^h>_{h,\mathbf{b},\Gamma_{12}^-}\\ &\qquad=\lambda<\boldsymbol{\sigma}_2{}^{n},\boldsymbol{\tau}^h>_{h,\mathbf{b},\Gamma_{12}^-}\quad\forall\boldsymbol{\tau}^h\in\boldsymbol{\Sigma}_1^h,\end{aligned}$$

$$\begin{aligned}&(\boldsymbol{\sigma}_1{}^{n+1},\mathbf{D}(\mathbf{v}^h))+2(1-\alpha)(\mathbf{D}(\mathbf{u}_1{}^{n+1}),\mathbf{D}(\mathbf{v}^h))-(p_1{}^{n+1},\nabla\cdot\mathbf{v}^h)+\frac{1}{\epsilon}(\mathbf{u}_1{}^{n+1},\mathbf{v}^h)_{\Gamma_0}\\ &\qquad=(\mathbf{f},\mathbf{v}^h)+\frac{1}{\epsilon}(\mathbf{u}_2{}^{n},\mathbf{v}^h)_{\Gamma_0}\quad\forall\mathbf{v}^h\in\mathbf{X}_1^h,\end{aligned}$$

$$(q^h,\nabla\cdot\mathbf{u}_1{}^{n+1})=0\quad\forall q\in S_1^h,$$

and

$$
\begin{aligned}
&(\boldsymbol{\sigma_2}^{n+1}, \boldsymbol{\tau}^h) + \lambda((\mathbf{b}\cdot\nabla)\boldsymbol{\sigma}_2^{n+1}, \boldsymbol{\tau}^h) + \lambda < \boldsymbol{\sigma}_2^{n+1^+} - \boldsymbol{\sigma}_2^{n+1^-}, \boldsymbol{\tau}^{h^+} >_{h,\mathbf{b}} \\
&\quad + \lambda\,(g_a(\boldsymbol{\sigma_2}^{n+1}, \nabla\mathbf{b}), \boldsymbol{\tau}^h) - 2\alpha(\mathbf{D}(\mathbf{u_2}^{n+1}), \boldsymbol{\tau}^h) + \lambda < \boldsymbol{\sigma_2}^{n+1}, \boldsymbol{\tau}^h >_{h,\mathbf{b},\Gamma_{21}^-} \\
&\qquad = \lambda < \boldsymbol{\sigma_1}^n, \boldsymbol{\tau}^h >_{h,\mathbf{b},\Gamma_{21}^-} \quad \forall \boldsymbol{\tau}^h \in \boldsymbol{\Sigma}_2^h\,,
\end{aligned}
$$

$$
\begin{aligned}
&(\boldsymbol{\sigma_2}^{n+1}, \mathbf{D}(\mathbf{v}^h)) + 2(1-\alpha)(\mathbf{D}(\mathbf{u_2}^{n+1}), \mathbf{D}(\mathbf{v}^h)) - (p_2{}^{n+1}, \nabla\cdot\mathbf{v}^h) + \frac{1}{\epsilon}(\mathbf{u_2}^{n+1}, \mathbf{v}^h)_{\Gamma_0} \\
&\qquad = (\mathbf{f}, \mathbf{v}^h) + \frac{1}{\epsilon}(\mathbf{u_1}^n, \mathbf{v}^h)_{\Gamma_0} \quad \forall \mathbf{v}^h \in \mathbf{X}_2^h\,,
\end{aligned}
$$

$$
(q^h, \nabla\cdot {\mathbf{u}_2^h}^{n+1}) = 0 \quad \forall q \in S_2^h\,.
$$

Note that the difference between two algorithms lies in the g_a terms of the constitutive equations. Convergence of Algorithm 5.3 can be shown under stronger conditions on parameters λM and α; we need the assumptions $\lambda M < 2\overline{C}^2 h(1-2\lambda M d)$ and $\frac{\lambda M}{2} - 4\tilde{C}^2\alpha(1-\alpha) < \frac{\alpha}{\epsilon} < \overline{C}^2 h(1-2\lambda M d)$, where $\overline{C}$ is the constant appears in the inverse inequality (5.13). The requirement on α is more restrictive than the condition in the Theorem 5.2 since ϵ is expected to be Ch^2 as stated in Remark 4.3. However, it turned out that the conditions for convergence are not necessary in numerical tests and the convergence rate of each algorithm is strongly dependent on ϵ. The convergence issue will be addressed in the next section.

.

6 Preliminary Numerical Results

In this section we present numerical results obtained using Algorithm 5.3. The example is a non-physical problem with domain $\Omega = [0,1]\times[0,1]$ and a specified solution. The function $\mathbf{b}$ was chosen to be the exact solution $\mathbf{u}$. The right hand side functions in (2.5)-(2.8) were appropriately given so that the exact solution was

$$
\begin{cases}
\mathbf{u} = \begin{bmatrix} \sin(\pi x) y(y-1) \\ \sin(x)\,(x-1) y \cos(\pi y/2) \end{bmatrix} \\
p = \cos(2\pi x)\, y(y-1) \\
\boldsymbol{\sigma} = 2\alpha\, D(\mathbf{u})
\end{cases} .
$$

The parameters λ, α and a in the equations were chosen as 1, 0.5 and 0, respectively. We used the Taylor-Hood element for $(\mathbf{u}, p)$ and discontinuous linear polynomials for $\boldsymbol{\sigma}$. Although we assumed that *div* $\mathbf{u} = 0$ for analysis, it turns out that the *div free* condition is not necessary for the numerical experiments. Note that the example we chose has the homogeneous boundary condition on $\partial\Omega$, but *div* $\mathbf{u} \neq 0$.

The domain Ω was divided into two subdoamins $\Omega_1 = [0,1/2]\times[0,1]$ and $\Omega_2 = [1/2,1]\times[0,1]$ For comparison of accuracy, the solution to (4.1)-(4.3) was first computed in each subdomain using the exact Neumann and Dirichlet boundary data on the interface Γ_0. See Table 1 for errors and convergence rates. Then we computed solutions by Algorithm 5.1 using different values of ϵ. Errors for $\mathbf{u}$ and $\boldsymbol{\sigma}$ are presented in Table 2. First, note that the rates presented in Table 1 are better than the theoretical result in (2.18), which is not unusual in simulation of viscoelastic fluid flows with a known exact solution [5, 6, 12]. Although the scaling $\epsilon = Ch^2$ is suggested for an optimal error estimate in (4.7), the actual optimal scaling may be $\epsilon = Ch^4$ based on the computed rates in Table 1, and this is partially verified by results in Table 2. For example, for $h_0 = 1/4$, ϵ_0 should be as small as 1/80 so that errors are similar to those errors in Table 1. When h is reduced by half ($h_1 = 1/8$), the ϵ value should be as small as $\epsilon_1 = 1/1300$ ($\approx 1/16$ of ϵ_0), as shown in Table 2. Therefore, we expect ϵ needs to be as small as 1/20480 when $h = 1/16$, which we could not check as the maximum number of iterations were exceeded.

h	$\|\mathbf{u} - \mathbf{u}_1^h\|_1$	rate	$\|\mathbf{u} - \mathbf{u}_2^h\|_1$	rate	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_1^h\|_0$	rate	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_2^h\|_0$	rate
1/4	$3.337 \cdot 10^{-2}$		$3.083 \cdot 10^{-2}$		$1.862 \cdot 10^{-2}$		$2.206 \cdot 10^{-2}$	
1/8	$7.629 \cdot 10^{-3}$	2.1	$7.232 \cdot 10^{-3}$	2.1	$4.570 \cdot 10^{-3}$	2.0	$5.065 \cdot 10^{-3}$	2.1
1/16	$1.836 \cdot 10^{-3}$	2.1	$1.776 \cdot 10^{-3}$	2.0	$1.187 \cdot 10^{-3}$	1.9	$1.265 \cdot 10^{-3}$	2.0

Table 1: Errors by exact interface data

h	ϵ	iter. no.	$\|\mathbf{u} - \mathbf{u}_1^h\|_1$	$\|\mathbf{u} - \mathbf{u}_2^h\|_1$	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_1^h\|_0$	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_2^h\|_0$
1/4	1/20	13	$4.778 \cdot 10^{-2}$	$4.580 \cdot 10^{-2}$	$2.558 \cdot 10^{-2}$	$2.401 \cdot 10^{-2}$
	1/40	26	$3.798 \cdot 10^{-2}$	$3.512 \cdot 10^{-2}$	$2.177 \cdot 10^{-2}$	$2.272 \cdot 10^{-2}$
	1/60	38	$3.497 \cdot 10^{-2}$	$3.200 \cdot 10^{-2}$	$2.059 \cdot 10^{-2}$	$2.236 \cdot 10^{-2}$
	1/80	48	$3.391 \cdot 10^{-2}$	$3.094 \cdot 10^{-2}$	$2.009 \cdot 10^{-2}$	$2.221 \cdot 10^{-2}$
	1/100	54	$3.398 \cdot 10^{-2}$	$3.084 \cdot 10^{-2}$	$1.985 \cdot 10^{-2}$	$2.215 \cdot 10^{-2}$
	1/150	77	$3.387 \cdot 10^{-2}$	$3.087 \cdot 10^{-2}$	$1.960 \cdot 10^{-2}$	$2.211 \cdot 10^{-2}$
	1/200	101	$3.392 \cdot 10^{-2}$	$3.109 \cdot 10^{-2}$	$1.952 \cdot 10^{-2}$	$2.221 \cdot 10^{-2}$
	1/500	257	$3.398 \cdot 10^{-2}$	$3.179 \cdot 10^{-2}$	$1.946 \cdot 10^{-2}$	$2.216 \cdot 10^{-2}$
1/8	1/20	14	$4.053 \cdot 10^{-2}$	$4.208 \cdot 10^{-2}$	$1.999 \cdot 10^{-2}$	$1.130 \cdot 10^{-2}$
	1/60	34	$2.184 \cdot 10^{-2}$	$2.090 \cdot 10^{-2}$	$1.018 \cdot 10^{-2}$	$6.999 \cdot 10^{-3}$
	1/100	66	$1.531 \cdot 10^{-2}$	$1.458 \cdot 10^{-2}$	$7.537 \cdot 10^{-3}$	$5.994 \cdot 10^{-3}$
	1/200	134	$1.089 \cdot 10^{-2}$	$9.982 \cdot 10^{-3}$	$5.675 \cdot 10^{-3}$	$5.391 \cdot 10^{-3}$
	1/400	298	$8.492 \cdot 10^{-3}$	$7.796 \cdot 10^{-3}$	$4.924 \cdot 10^{-3}$	$5.162 \cdot 10^{-3}$
	1/500	404	$7.996 \cdot 10^{-3}$	$7.412 \cdot 10^{-3}$	$4.812 \cdot 10^{-3}$	$5.127 \cdot 10^{-3}$
	1/1000	704	$7.997 \cdot 10^{-3}$	$7.282 \cdot 10^{-3}$	$4.687 \cdot 10^{-3}$	$5.103 \cdot 10^{-3}$
	1/1300	973	$7.696 \cdot 10^{-3}$	$7.122 \cdot 10^{-3}$	$4.655 \cdot 10^{-3}$	$5.090 \cdot 10^{-3}$
1/16	1/10000	8000	$2.414 \cdot 10^{-3}$	$2.063 \cdot 10^{-3}$	$1.238 \cdot 10^{-3}$	$1.297 \cdot 10^{-3}$
	1/20000	15000	$2.830 \cdot 10^{-3}$	$2.310 \cdot 10^{-3}$	$1.274 \cdot 10^{-3}$	$1.322 \cdot 10^{-3}$

Table 2: Errors by various ϵ values

7 Conclusions and Future Work

We investigated a non-overlapping domain decomposition algorithm for the linearized viscoelastic fluid equations. The domain decomposition algorithm is presented as a discrete variational formulation for which we analyzed for the existence and uniqueness of a solution. Convergence of the domain decomposition solution both with respect to the grid size h and with respect to the parameter ϵ was also proved. We also presented some preliminary computational results. However, to make the method more practical, further studies are needed, mostly in the realm of efficient implementation. For example, convergence of algorithm 5.1 could be accelerated by introducing a relaxation parameter in the boundary integrals of the momentum equations as shown in [18]. A similar technique may be effective in handling stress boundary conditions on the interface. We will also study domain decomposition methods designed for viscoelastic fluid flows with other type of constitutive equations, for instance, fluids having a power law constitutive equation.

References

[1] F.P.T. Baaijens, Mixed finite element methods for viscoelastic flow analysis: A review, *J. Non-Newtonian Fluid Mech.* 79 (1998), pp. 361-385.

[2] J. Baranger and D. Sandri, Finite element approximation of viscoelastic fluid flow: Existence of approximate solutions and error bounds, *Numer. Math.* 63 (1992), pp. 13-27.

[3] S. Brenner and L. Scott, *The mathematical theory of finite element methods*, Springer-Verlag, 1994.

[4] A. E. Caola, Y. L. Joo, R. C. Armstrong and R. A. Brown, Highly parallel time integration of viscoelastic flows , *J. Non-Newtonian Fluid Mech.* 100, pp. 191-216, 2001.

[5] V.J. Ervin, H. Lee, and L.N. Ntasin, Analysis of the Oseen-viscoelastic fluid flow problem, to appear in *J. Non-Newtonian Fluid Mech.*

[6] V.J. Ervin and H. Lee, Defect correction method for viscoelastic fluid flows at high Weissenberg number, *Numerical Methods for PDEs*, 22 (2006), pp. 145-164.

[7] M. Fortin and A. Fortin, A new finite approach for the F.E.M. simulation of viscoelastic flows, *J. Non-Newtonian Fluid Mech.* 32 (1989), pp. 295-310.

[8] F. Gastaldi and L. Gastaldi and A. Quarteroni, Adaptive domain decomposition methods for advection dominated equation, *East-West J. Numer. Math.*, 4 (1996), pp. 165–206.

[9] V. Girault and P. Raviart, *Finite element methods for Navier-Stokes equations*, Springer, Berlin, 1986.

[10] M. Gunzburger and H. Lee, An optimization based domain decomposition method for the Navier-Stokes equations, *SIAM J. Num. Anal.* 37 (2000), pp. 1455-1480.

[11] P. Houston, C. Schwab and E. S*ddot*uli, Discontinuous *hp*-finite element methods for advective-diffusion-reaction problems, *SIAM J. Numer. Anal.* 39 (2002), pp. 2133-2163.

[12] H. Lee, A multigrid method for viscoelastic fluid flow, *SIAM J. Num. Anal.* 42 (2004), pp. 109-129.

[13] G. Lube and L. Muller and F.C. Otto,A non-overlapping domain decomposition method for stabilized finite element approximations of the Oseen equations, *J. Comput. Appl. Math.* 132 (2001), pp. 211–236.

[14] J.M. Marchal and M.J. Crochet, A new finite element for calculating viscoelastic flow, *J. Non-Newtonian Fluid Mech.* 26 (1987), pp. 77-114.

[15] A. Quarteroni and A. Valli, *Domain Decomposition Methods for Partial Differential Equations*, Oxford Science Publications, 1999.

[16] D.G. Radu, M. Normandin and J.R. Clearmont, A numerical approach for computing flows by local transformations and domain decomposition using an optimization algorithm, *Comput. Mehtods Appl. Mech. Engrg.* 191 (2002), pp. 4401-4419.

[17] T.C. Rebollo and E.C. Vera, Study of a non-overlapping domain decomposition method: Poisson and Stokes problems, *Appl. Numer. Math.*, 48, (2004), pp. 169–194.

[18] T.C. Rebollo and E.C. Vera, Study of a non-overlapping domain decomposition method: Steady Navier-Stokes equations, *Appl. Numer. Math.*, 55, (2005), pp. 100–124.

[19] M. Renardy, Existence of slow steady flows of viscoelastic fluids with differential constitutive equations, *Z.A.M.M.* 65 (1985), pp. 449-451.

[20] D. Sandri, Finite element approximation of viscoelastic fluid flow: Existence of approximate solutions and error bounds. Continuous approximation of the stress, *SIAM J. Numer. Anal.* 31 (1994), pp. 362-377.

[21] C. Schwab, *p- and hp- Finite element methods*, Theory and applications to solid and fluid mechanics, Oxford University Press, 1998.

[22] B. Smith and P. Bjørstad and W. Gropp, *Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations*, Cambridge University Press, 1996.

[23] A. Souvaliotis and A.N. Beris, Spectral colloacation/domain decomposition method for viscoelastic flow simulations in model porous geometries, *Comput. Mehtods Appl. Mech. Engrg.* 129 (1996), pp. 9-28.