The Rank One Abelian Stark Conjecture:

Stark's conjectures relate values of derivatives of
partial zeta functions at $s=0$

to
logarithms of absolute values of units in algebraic
number fields.

- Analogous to (and a refinement of) the Dirichlet
class number formula.

- A version for units of the BSD Conjecture for elliptic
curves. (In fact, both can be simultaneously
generalized and refined in the Eichler-Shimura
Number Conjecture.)

- "Abelian" indicates we will consider extensions $K/F$
that are Galois with abelian Galois groups.

- "Rank one" means we'll consider $1^{st}$ derivatives of
$g$-functions (conjecture new precise here)

- Rubin later made a similar precise conjecture in
the higher rank case.

Example: $f \in \mathbb{Z}_{\mathbb{Z}}$, $a \in \mathbb{Z}$ ($a, A = 1$).

Define $\zeta_f(n) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $s \in \mathbb{C}$

$\text{Re}(s) > 1$
This is essentially the Hurwitz zeta function.

\[ S_{\text{Hur}}(x,s) = \sum_{n \geq 0} \frac{1}{(x+n)^s} \quad x/s \in \mathbb{C}, \quad \Re(x) > 0, \quad \Re(s) > 1. \]

\[ S_f(a,s) = f^{-s} \int_0^a \frac{1}{s-1} + b \gamma(f) + \cdots + \frac{1}{s-1} + \cdots \]

Stark's conjecture concerns \( b(a,f) \).

Using the functional equation we can move from \( s=1 \) to \( s=0 \).

Define \( S_f^+(a,s) = S_f(a,s) + S_f(-a,s) \).

For \( 0 < a < f \), we have

\[ S_f(a,0) = \frac{1}{2} - \frac{a}{f}. \]

\[ \Rightarrow S_f^+(a,0) = 0. \]

Also

\[ S_f^+(a,s) = C(a,f) s + \cdots \]

\[ \frac{d}{ds} S_{\text{Hur}}(x,s) \bigg|_{s=0} = \log \Gamma(x) - \frac{1}{2} \log(2\pi) \]

\[ \Rightarrow C(a,f) = \log \left( \frac{\Gamma(f) \Gamma(1-f)}{2\pi} \right) \]
\[= -\log(2 \sin \left(\frac{\pi}{q}\right))\]
\[= -\frac{1}{2} \log(2 - 2 \cos \left(\frac{\pi \alpha}{q}\right))\]
\[= -\frac{1}{2} \log(u_{q, \alpha}).\]

\[u_{q, \alpha} = (1 - \frac{\pi}{q})(1 - \frac{\pi}{q})\] where \(\pi_f = e^{2\pi i \rho_f}\)

Thus, we have \(u_{q, \alpha} \leq K = \Theta(\pi_f^+) = \Theta(\pi_f^+ \pi_f^-)\).

**Exercise:** \(u_{q, \alpha} \in \mathcal{O}_k\left[\frac{1}{q}\right]^x\)

In fact, \(u_{q, \alpha} \leq \mathcal{O}_k^x\) if \(f\) is divisible by at least 2 distinct primes.

**Summary:**
\[\pi_f^+ (a, 0) = 0\]
and
\[(\pi_f^+)^y (a, 0) = -\frac{1}{2} \log u_{q, \alpha}\]

where
\[u_{q, \alpha} \in \begin{cases} \mathcal{O}_k^x & \text{if div. by at least 2 primes} \\ \Theta_k(\pi_f^+)^y & \text{if prime power} \end{cases}\]

where \(K = \Theta(\pi_f^+)\).

**The Conjecture:**

\(K/F\) an abelian extension of number fields.
$S = \text{finite set of places of } F$

$2 \geq \sum \nu \in \nu^S \cap \text{finite places that ramify in } K.$

Assume $|S| \geq 3$. ($|S| = 2$ is ok, see notes)

$\mathcal{S}_k F^\sigma = \sum (N_n)^{-s} \quad (s \in \mathbb{C}, \text{ Re}(s) > 1).$

$(\mathcal{N}, s) = 1, \sigma_n = 0$

$\sigma \in \text{Gal}(K/F) \Rightarrow$ end of Galois case. To $\mathcal{N}$ via CRT.

$S = \mathcal{O}_F, \mathcal{P} \not\in \mathcal{P}_S, \mathcal{S}_k F^\sigma(s, \sigma) = \delta_{\mathcal{P}, (a, s)}$

$\mathcal{O}_F = \mathcal{O}_F^\sigma$, $\mathcal{P} \not\in \mathcal{P}_S, \mathcal{S}_k F^\sigma(s, \sigma) = \delta_{\mathcal{P}, (a, s)}$

Let $U_v = U_v(K) = \mathcal{O}_K^\sigma: \mathcal{I}_v = 1 \quad \forall w, v \neq w$.

where $v$ is a place of $F$.

"Strong $v$-units" since we allow $w$ to be arch.

Fix $v \in S$. Assume $v$ splits completely in $K$. Let $w$

be a place of $K$ above $v$.

$v$ splits in $K \Rightarrow \mathcal{S}_k F^\sigma(s, \sigma) = 0$

Conjecture: $\exists w \in U_v(K) \text{ s.t. }$

$\mathcal{S}_k F^\sigma(s, \sigma) = -\frac{1}{2} \log |u^\sigma w| \quad \forall \sigma \in \text{Gal}(K/F)$

$e = \# \mu_v(K)$, where $u$ depends on $w$. Furthermore,

$K(u^{1/e})/F$ is abelian.

Note: $|u^\sigma w|$ is specified $\forall w$, $w \neq K$. 
\[ T = \mathfrak{p} \mathfrak{g} \text{ prime ideal } \mathfrak{p} \subseteq \mathfrak{g}, \mathfrak{q} \subseteq S \]

\[ \text{Char} \left( \mathcal{O}_F/v \right) = k \geq \left[ F : \mathcal{O} \right] + 2. \]

\[ \Delta_{K,F,S}(\sigma, \delta) = \Delta_{K,F,S}(\sigma, \delta) - N\sigma^{-1} \Delta_{K,F,S}(\sigma^{-1}, \delta) \]

Equivalent Conjs: \[ \exists U_T \in U_{v,T} = \left\{ u \in U_v : u \equiv (\text{mod } \mathcal{O}_v) \right\} \]

\[ \text{s.t.} \]

\[ \Delta_{K,F,S}(\sigma, 0) = -\log |U_T \omega| \]

\[ u, U_T \text{ related: } \lambda = u^{1/e}, U_T = \lambda^{1/(e)} \]

\[ U_T \text{ unique.} \]

If \( S \) contains \( v, v' \) that split completely in \( K \), then

\[ \Delta_{K,F}(\sigma, 0) = 0. \Rightarrow u = 1 \text{ works in conjecture. So we only consider } \]

- Case TR_0: \( F = \) totally real, \( v = \) real, places \( v \)

\( K \) above \( v \) are real, all other arch. places are complex.

- Case AR: \( F = \) "almost totally real", i.e., \( F \)

has exactly 1 complex place \( v \), \( K \) = totally complex.

- Case TRp: \( F = \) totally real, \( v = \) finite place,

\( K \) = totally complex.

**Case TR_0:**

\[ u^e = \exp^{-2\Delta_{K,F,S}(\sigma, 0)} \text{ by inverting the formula} \]

This can be viewed as progress towards Hilbert's
12th problem

Case ATR:

\[ |\psi|_w = e^{-\frac{5}{4}\pi s^2 + (0,0)} \]

But we don't see what \( \psi \) is in \( \mathbb{C} \), only its absolute value. What is \( \arg(\psi) \)?

Motivating Question: Can we give an exact formula for \( \psi \in \mathbb{C} \) and not just its absolute value?

ATR: Rene, Siegelh, Chandha, Daron, yes!

TRp: Daron - Diaaphtha

Chapelaime?

Yes!

Diaaphtha

Chandha - Chandha

5th: \( \text{5-functions are not the whole story on the analytic side. 5-functions are merely shadows of "high" or "more refined" objects (cohomology classes, Arthurian zero-functions).} \)

Main obstruction: units in ground field \( F \).

Case TRp: \( v = \frac{p}{2} \in \mathcal{O}_F \), prime ideal.

Let \( R = S - \frac{p}{2} \mathfrak{p} \), \( \omega = \beta_{1, \mathfrak{p}} \),.
$$\hat{S}_{K/F,S,T}(\sigma, s) = (1 - N_{g}^{-s}) \hat{S}_{K/F,R,T}(\sigma, s)$$

$$\hat{S}_{K/F,S,T}(\sigma, 0) = (\log N_{g}) \hat{S}_{K/F,R,T}(\sigma, 0).$$

$$- \log |U_{F}^{T}|_{B} = (\log N_{g}) \text{ord}_{B}(U_{F})$$

Shank's conjecture in this case says:

If $$U_{F} \in U_{F,T}$$ s.t. $$\text{ord}_{B}(U_{F}) = \hat{3}_{K/F,R,T}(\sigma, 0).$$

Note: RHS is in $$\mathbb{Z}$$ by work of Deligne-Ribet / Pi. Cassou-Nogues / Barsky.