Special values modulo $p$:

Non-vanishing of values:

$\zeta(5) = \text{Riemann zeta function}$

$\zeta(k) \equiv \infty \quad \text{if} \quad k \text{ not odd}$

$\zeta(-1) = -\frac{1}{12}, \quad \zeta(-2) = \frac{1}{60}$

$\zeta(1-2k) = -B_{2k}/k$ \quad \text{where} \quad \frac{1}{e^x-1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}$

$\text{If} \quad p \text{ odd prime}$

Kummer: \quad $h(Q(\sqrt[p]{p}))$ is divisible by $p \iff p \mid \text{num}(B_k)$

$k = 2, 4, 6, \ldots, p-3$

$N \geq 2$ integer, $L(s, x)$ is a Dirichlet $L$-function, $x$ prime mod $N$.

$L(1-n, x) = -\frac{B_n x}{n}$ \quad \text{when} \quad \sum_{a=1}^{\infty} \frac{x(a) e^{at}}{e^{at} - 1} = \sum_{a=1}^{\infty} B_n x \frac{t^n}{n!}$

$h(Q(\sqrt[p]{p})) = \frac{\sqrt[p]{p}}{k} \quad L(1, x) \sim B_2 x$

$x \in S$ = some family

How often is $B_n x$ divisible by some fixed prime or $Q^\infty$ (case: char $k$)

$a \circ x$ varies?

Example: $S = \text{quadratic, char} \circ \text{cyclic}(p)$

$p$-adic families:

$S = \text{Dirichlet class, cyclic, } p^n \text{ as } n \to \infty$
If \( e_n \) is the exponent of \( p \) in class number of \( \mathbb{Q}(\sqrt{5p^n}) \),

then \( e_n = \lambda n + \mu p^n + \nu \) for all \( n \geq 0 \), \( \lambda, \mu, \nu \in \mathbb{Z} \).

**Class num:** \( \mu = 0 \).

Exponent of \( l \neq p \) in \( h(\mathbb{Q}(\sqrt{5p})) \)?

The evidence was that \( \text{ord}_l(h(\mathbb{Q}(\sqrt{5p}))) < C_{l, p} \).


**Starting point:** Explicit formulas for class numbers in terms of Bernoulli numbers.

**Formula (class num):**

\[ B_n \sim \text{related to digit } (p \text{-adic)} \text{ or } p \text{-1 roots of unity.} \]

**Point:** prove that digit of \( p \)-1 root of unity behave like indep. random variables, first hint or probability...

Recall: \( a_0 + a_1 p + \cdots = x \in \mathbb{Z}_p \) is called *normal* if every string of length \( k \) of digits appears with frequency \( p^{-k} \).

Easy to show set of non-normal element has measure 0, but very hard to determine if a specific one is normal or not.
Suppose \( Y_1, \ldots, Y_r \) are linearly indep. over \( \mathbb{Q} \). Then for almost all \( \beta \in \mathbb{Z}_p \), the sequence of \( \beta \)-tuples \( \langle x_1, \ldots, x_r \rangle \) is unif. distributed in \( (0,1)^r \),

\[
X_\alpha (x) = \frac{\text{unique integer in } \Gamma_0, p^n}{p^n},
\]

Analogy: (Kneser) Suppose \( Y_1, \ldots, Y_r \in \mathbb{R} \) are linearly indep. over \( \mathbb{Q} \), then \( (tY_1, tY_2, \ldots, tY_r) \) has dense image (unif.) in \( (0,1)^r \).

This was proved by Sinnott. He used Euler's formula that relates \( \zeta(n) \) to derivatives of rational functions.

\[
\mathbb{F} (T^{-1}) = \text{Laurent series in } T^{-1}. \quad (\text{over } \mathbb{F} = \overline{\mathbb{F}_p})
\]

Sinnott: Suppose \( Y_1, \ldots, Y_r \in \mathbb{Z}_p \) are linearly indep. over \( \mathbb{Q} \), then

\[
T^{Y_1}, \ldots, T^{Y_r} \text{ are algebraically indep. in } \mathbb{F} (T^{-1})^d,
\]

\[
T^x = \sum (\binom{x}{n} (T^{-1})^n)
\]

This seems at first glance to be an entirely algebraic proof.

\( \mathbb{F} (T^{-1}) \) power series in 1 variable, formal completion of \( \mathbb{F}_p \).

\[
\mathbb{F} (T^{Y_1}, T^{Y_2}, \ldots, T^{Y_r}) \subset \mathbb{F} (T^{-1})
\]

poly ring is

r variables.
Geometrically: 1 parameter formal $G$ dense inside $r$-dim space.

As it is actually similar to before, it seems all examples, at least one looked at today, are something small mapping into something large and the image turns out to be dense.

Example: Hecke L- functions (Hida)

Key ingredient: Chai

$k = \text{alg. closed field of char} \neq 0$, $X = \text{smooth, finite dim. formal p-div. group over } k$

$E_k = \text{End}(X)$, $E = E_k \otimes \mathbb{Q}_p = \text{finite dim. s.s. over } \mathbb{Q}_p$.

$E = \text{dim. alg. rep. over } \mathbb{Q}_p$ s.t. $E(R) = (E \otimes \mathbb{Q}_p R)^\times$, $R$ any $\mathbb{Q}_p$- alg.

$G$ any alg. group over $\mathbb{Q}_p$ and $\rho: G \to E$ in a homom., can regard $\rho$ as a rep. of $G$ on $E$ with $E < \text{Aut}(E)$.

Chai: Suppose that the induced rep. is not a subquotient of $\rho$ (of $G$ on $E$). Suppose $Z$ is a reduced, irreducible, closed formal subscheme of $X$ closed under action of an open subgroup of $G(\mathbb{Q}_p)$. Then $Z$ is closed under group law of $X$ and is a p-div. subgroup.

Anticyclotomic twist of $G_b$ L-functions:

$F = \text{totally real field}, K/F = \text{imag. quad ext.}$
Hilbert zeta form \( \zeta_L(s) \) at \( s = \frac{1}{2} \) for \( F \).

Look at \( L(x, x, s) \), \( x \) anticyclic character of \( \mathbb{A}_F^* \).

Means: \( x \) factors through \( \mathbb{A}_F^* \) and \( \mathbb{A}_K^* \).

Central char. of \( x \) is unram. and \( xw = 1 \) on \( \mathbb{A}_F^* \).

Look at \( L(x, x, \frac{1}{2}) \) for \( x \) running over anticyclic char.

of \( \mathbb{A}_K^* \), \( p^s \to 0 \) as \( s \to \infty \) where \( p = \text{prime of } K \).

Expect \( L(x, x, \frac{1}{2}) \neq 0 \) to vanish to order 1 (typically).

Proved under mild assumptions (Conjectures). Don't want to spend time on the statement, talk on the ingredients that go into the proof, and the analogies with stuff already discussed.

Key ingredient: Thm of M. Ratner

\[ G = SL_2(\mathbb{Q}_p), \ G_i \subset G \text{ discrete, cocompact subgroups. Say that } \]

\( G_i \) and \( G_j \) are commensurable if \( G_i \cap G_j \) has finite index in both.

Thm (Ratner): Suppose that \( G_i \) and \( G_j \) are not commensurable,

Then \( G_i \cdot G_j = \{ \gamma_i \gamma_j : \gamma_i \in G_i, \gamma_j \in G_j \} \) is dense in \( G \),

\( \not \) a group!

Kneser: \( G = \mathbb{R}, \ G_i, G_j \) discrete subgroups \( \mathbb{Z} Y_i, \mathbb{Z} Y_j, \ G_i \cap G_j \)

is dense iff \( Y_1, Y_2 \) an indep over \( \mathbb{Q} \), \( G_i \cap G_j \) is trivial

(most commensurable).
More generally, suppose $\Gamma_1, \Gamma_2, \ldots, \Gamma_r$ are pairwise not comm., then
the image of the diagonal inside

$$X = \Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_r \backslash G^r$$

in dense in $X$.

More generally still

$G$ = any $p$-adic Lie group

$\Gamma$ = discrete subgroup s.t. $\Gamma \backslash G$ has finite measure. Let the
unique $G$-invariant Borel measure.

$H \subset G$ any subgroup generated by image of mon.

$U: (\mathbb{Q}_p) \rightarrow G$

then the closure $\overline{H}$ of $H$ in $\Gamma \backslash G$ in having $\exists H' \subset H$

s.t. $\Gamma \cdot (H') = \overline{H}$. 