On l-adic families of ramified representations of GL_2(Q_p):

This is currently work in progress. There is substantial overlap w/ indp.
work of Emerton. The notation is unfortunately not the same!

Motivation: Passage from f to ft "decrease wall in families."

- Congruence
- R=T theorem
- p-adic modular forms

The representation theory side has been mostly done on fields. One
would like to be able to do this in the context so as to be able to
study families as well. The global case seems to be hopeless, but
one can try for local Langlands a work in families.

\[ \begin{array}{c}
\text{mod.} \\
\text{Admissible reps.} \\
of \text{GL}_n(\mathbb{Q}_p)/\mathbb{F}_p
\end{array} \begin{array}{c}
\text{bij.} \\
\text{Frob. s.s. Weil-Deligne reps.}
\end{array} \begin{array}{c}
\text{WD}_{\mathbb{Q}_p} \rightarrow \text{GL}_n(\mathbb{C})
\end{array} \]

- Alex Paulin: families of adn reps. one eigenvalue
- Matthew Emerton: see his talk.

Starting point for this work is

\[ \begin{array}{c}
\text{inad. adn. reps.} \\
of \text{GL}_n(\mathbb{Q}_p)/\mathbb{F}_p
\end{array} \begin{array}{c}
\text{bij.} \\
\text{n-dim. Weil-Deligne reps. over } \mathbb{F}_p
\end{array} \begin{array}{c}
\text{Supercuspidal} \\
\text{inv.}
\end{array} \begin{array}{c}
\text{inad. reps.}
\end{array} \]

\text{I odd, } \ell \neq p.
Given \( \bar{\pi} : G_{\mathbb{A}} \to GL_3(\overline{\mathbb{F}_p}) \), there is a corresponding \( \pi \). Fix a finite length \( \mathbb{F}_p \)-linear \( \mathbb{W}(\overline{\mathbb{F}_p}) \)-algebra \( A \).

**Def:** 1. An \( A \)-deformation \( \pi \rightarrow \overline{\pi} \) is an \( A[GL_3(\mathbb{A}_f)] \)-module, free over \( A \), admissible (\( \pi = \text{dim} \pi^u \), \( \pi^u \) is free from some \( u \)),

and an isomorphism \( A_{\mathbb{A}_m} \otimes \overline{\pi} \rightarrow \overline{\pi} \).

2. An \( A \)-deformation \( \pi \rightarrow \overline{\pi} \) is \( \rho : G_{\mathbb{A}} \to GL_3(A) \) such that \( \rho_{\mathbb{A}_m} \otimes \overline{\pi} \rightarrow \overline{\rho} \).

**Thm:** Def \( \overline{\pi} \) fixed, there is a natural bijection

\[
\{ A \text{-degs. } \pi \text{ of } \overline{\pi} \} \leftrightarrow \{ \overline{A} \text{-degs. } \rho \text{ of } \overline{\rho} \}
\]

There is a natural isomorphism

\[
\mathbb{R}^{\text{reg}}_{\overline{\pi}} \leftrightarrow \mathbb{R}^{\text{reg}}_{\overline{\rho}}
\]

Uniquely characterized by inducing usual char. of \( L \cdot L.C \) on \( \overline{Q}_L \)-alg.

**Pf:** Compute both sides. Use explicit L.I.C.

\( \chi : E \to \overline{\mathbb{F}_p} \) quadratic

**Case (1):** \( \bar{\rho} \) primitive, i.e., not induced from a quadratic char. (\( p \neq 2 \)).

\[
\text{def } \chi \bar{\rho} = \text{def } \chi \bar{\rho}
\]

\[
\text{def } \chi \overline{\pi} = \text{def } \chi \overline{\rho}
\]

**Case (a):** \( E/\mathbb{Q}_L \) quadratic (\( \tau = \text{conj} \))

\[
\left\{ \text{char } \epsilon : E \to \overline{\mathbb{F}_p} \right\} \leftrightarrow \left\{ \epsilon : E \to \overline{\mathbb{F}_p} \right\}
\]

\[
\downarrow
\]

\[\overline{\pi} \]

\[
\uparrow \text{ twist to make } \mathbb{F}_p \text{-form match}
\]

\[
\text{type } T \in GL_3(\mathbb{A}_f), \text{ f.d. rep of } \mathbb{F}_p
\]
What if $p$ is conditional? Take $p = \{\mathcal{G}, \mathcal{G}(\mathcal{F})\}$, and note:

$p = \{\mathcal{G}, \mathcal{G}(\mathcal{F})\}$

Then $p$ is still true. We can also consider the case where $p$ is not true and see what happens. $p = \{\mathcal{G}, \mathcal{G}(\mathcal{F})\}$

$p$ is not equivalent to $p$. This is because $p$ is true in some cases but not in others.
\[ \rho^\text{ver}(\sigma) = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \]

\[ \rho \text{ sim.} \quad \text{2-dim} \]

\[ N \text{ Steinberg, } \text{1-dim} \]

As there really is no hope here, we need to lower our expectations.

\[ R^\text{ver}_p \subseteq W(F) \otimes_k W(F)^* \text{ over } W(F_k) \otimes_k W(F)^* \quad \text{a } \text{principal series, generic quotient over } Z(p) \]

\[ N \text{ Steinberg we chose } X(F) = N \text{ over } W(F_k) \otimes_k W(F)^* \]

\[ f: M_{(\alpha-p)} M \rightarrow N \text{ for in } N \text{ of } \text{N} \text{/N} \]

\[ \text{pair } (m, n) \in M \otimes N \text{ for in } N \text{/N} \]

\[ \text{Thm: } \text{At } 2 \text{ eigenform of level } N, \text{ pN}, \text{ then} \]

\[ \rho_p |_{p \Gamma N} = 1 \otimes w \]

\[ S = \lim_{\rightarrow} S_2(\Gamma(p)|\Gamma(1)) W(F) \]

\[ M \otimes \Pi \text{ corresponding to } g \in \mathfrak{g} \text{ in } \mathfrak{g}_L \text{ GL}_2(\mathcal{O}_p) \text{ - module} \]

\[ \text{Thm (Emerton): } A \text{ a reduced complete Noetherian local flat } W(F) \text{ - } \]

\[ \text{algebra, } \rho: \mathfrak{g}_p \rightarrow \mathfrak{gl}_2(A). \text{ Then } \exists \text{ at most one} \]

\[ A[GL_2(\mathcal{O}_p)] \text{ - module } \Pi \text{ st.} \]

1) \( \Pi \) is "A-torsion free" (every associated prime of \( \Pi \) is minimal)

2) At minimal prime \( \mathfrak{p} \), \( \Pi \) corresponds to \( \rho \) via L.I.C.
\textbf{Conjecture (Emorito):} There always is an $a$-module.

It seems feasible that the construction given here will give this conjecture.