Moduli stacks of local Hecke representations.

Joint project w/ Toby Gee

Idea: To make moduli spaces of p-odori & mod-p reps. of $G_k$ where $k/p$ is finite.

Murae, et. al.: Formal moduli of such reps. We want actual algebraic moduli. (as p really varies in families.)

One motivation: Reduction of crystalline Hecke representations.

If $k = \mathbb{Q}_p$, $p$: $G_{\mathbb{Q}_p} \to GL_2(\mathbb{Q}_p)$ etc. and reps.

If H.T weights $0$, $k - 1$, dot = cycle $1 - k$.

$p$ is (essentially) classified by $a_P = h$. of crystalline Frob.

Formed $\mathcal{M}/\mathbb{Z}_p$.

Red. mod $a_P \to \overline{a_P}$.

$2 \leq k \leq p$

$a_P \to \overline{a_P}$.

$k = p + 1$

coordinate here in $(a_P/p)$.

Inside are $\gamma$ (the cycle $1 - m$) $\mapsto 5^{m-1}F^2$

Formed $\mathcal{M}/\mathbb{Z}_p$ blow up at a pt in the special fibre.
Goal: For any $K/Q_p$ finite, type $r$, integer $h$, to construct a formal abelian stack $\mathcal{X}_r^h$ (dep. fin. type).

\[ \mathcal{X}_r^h \]

\[ \Sigma \]

$\mathbb{F}_p - \text{pts} \leftrightarrow p : G_K \to G_{L_n}(\mathbb{F}_p)$ s.t.

$p \otimes \rho \in \text{pol. semi-stable}$

$k \in \{0, h\}$ and type $r$.

And also construct $\mathcal{X}$ a finite type alg. stack $/\mathbb{F}_p$ s.t.

$\mathbb{F}_p - \text{pts of } \mathcal{X} \leftrightarrow \overline{\rho} : G_K \to G_{L_n}(\mathbb{F}_p)$ and
gives an embedding $\mathcal{X}_{\mathbb{F}_p} \hookrightarrow \mathcal{X}$ whose image is a union of components s.t. the specialization map

$\mathcal{X}_{\mathbb{F}_p}(\mathbb{F}_p) \to \mathcal{X}_{\text{red}}(\mathbb{F}_p) \hookrightarrow \mathcal{X}(\mathbb{F}_p)$

in $\rho \mapsto \overline{\rho}$.

and $K/Q_p$ unram.

If $h$ is small, one can make component of $\mathcal{X}$ as moduli space of $\mathbb{F}_p$-modules. (Daniel le Thése)

If $h$ isn't small, this doesn't work.

If $K = Q_p$, $G_{L_n}$, $h = p-1$: fix det trivial, then the reduction $\overline{\rho}$ are in the representation family

\[
\begin{pmatrix}
0 & * \\
0 & \text{unram}_p
\end{pmatrix}
\]

So we have the following picture:
F-L. Theory:

produce,

so the functor to Deloc reps is not faithful.

As we need to glue them somehow along this line.

We have good progress towards the construction of $\mathbb{A}, \mathbb{A}$, etc.
The details are worked out so far after replacing $K$ by $K_{\infty}$, where $K_{\infty} = \bigcup_{n \geq 1} K(\pi_n)$.

This is enough to handle the case of $\mathbb{A}_l$.

To get the general case, we will have to add $\overline{\mathbb{Q}}$-structure

following Tony Liu.

Once $K_{\infty}$, Fontaine-Berger-Kisin give a nice integral $\mathbb{Q}$-adic

Hodge theory, which Pappas-Rapoport put in moduli.

Work over $\mathbb{Z}/p^2$, a fixed. Fix $d$.

$$\mathbb{Z}_{(p)} \to \mathbb{R}$$

$M \to M[1]$.

Moduli stack of $K$-modules

higher $\mathbb{A}$.

A given $\mathbb{Z}/p$-et locally free $\mathbb{Q}_d$.

$M / \mathbb{A}_{et} \otimes \mathbb{Q}_d$.
$G^m \rightarrow M$

cornered killed by $E_{sh}$.

Build $\mathcal{F}$ as the union of the image of these maps.

We need to construct these images as objects in alg. geo.

杈の more they exist at a point in a case by case basis

What we have is

\[ G^m \rightarrow \mathcal{R} \] is proper

\[ \mathcal{R} \] is a stack

\[ \mathcal{R} \] is huge, $\Delta : \mathcal{R} \rightarrow \mathcal{R} \times \mathcal{R}$ is representable, finite type.

of fppf type / $\mathbb{Z}/p^2$

$\mathcal{R}$ admits a neutral map of each fppf pt.

Set-Disp: $\mathcal{F} \xrightarrow{\mathcal{S}} \mathcal{F}$ stack / Set

alg. stack $\mathcal{R}$, $\Delta : \mathcal{R} \rightarrow \mathcal{R} \times \mathcal{R}$ is rep. by alg. space

aff. for

$G$ has neutral map at each fppf

Define: "scheme-theoretic ring" of $\mathcal{S}$, $\mathcal{Z} \rightarrow \mathcal{F}$ substack

local $f$ is

* of $A$ is "Artinian", $\mathcal{Z}(A) \subseteq \mathcal{F}(A)$ and the maps $\mathcal{F}$

Spec $A \rightarrow \mathcal{F}$ which factor as $\text{Spec } A \rightarrow \text{Spec } B \rightarrow \mathcal{F}$ where

- $B$ is complete local $\mathbb{N}$eth / $\mathcal{S}$.
- $\mathcal{F}$ is $\text{sch } \mathcal{S}$, $\mathcal{F}$-constant

Spec $B$

* of $T$ is finite type, $\mathcal{Z}(T) \rightarrow \mathcal{F}(T)$ to be maps

s.t. $\text{Spec } \mathcal{O}_T/\mathfrak{m}_T \rightarrow \mathcal{F}$ lie in $\mathcal{Z}$

\[ A \otimes T \] all $T$ a finite type pt in $T$
of $T/S$ is affine, write $T = \lim T_i$ and define $Z(T) = \bigcup_i Z(T_i) \rightarrow T_0 \rightarrow T(1)$.

**Thm:** If $\pi : X \rightarrow T$ is proper, and $X$ is an alg.

- Stack of fin. type $/S$, $\Delta : T \rightarrow T \times T$ is repre. by alg. space $X$.
- $T$ admits versal ring at each fin. pt.
- Then $Z$ admits effective versal ring at fin. pt., then $Z$ is an alg. stack $\text{fin} \times \text{type} / S$. 