On Saito-Kurokawa Lifts

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Recall

\[ \iota_n = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}. \]

\[ \text{GSp}_{2n} = \{ g \in \text{GL}_{2n} : {}^t g \iota_n g = \mu_n(g) \iota_n, \quad \mu_n(g) \in \text{GL}_1 \} \]

and

\[ \text{Sp}_{2n} = \ker(\mu_n) \]

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and

$$\text{Sp}_{2n} = \ker(\mu_n)$$

Siegel upper half space of degree $n$ is given by

$$\mathcal{H}^n = \{ Z \in \text{Mat}_n(\mathbb{C}) : {}^t Z = Z, \text{Im}(Z) > 0 \}.$$
Action of $\text{GSp}_{2n}^+(\mathbb{R})$ on $\mathfrak{h}^n$

$\text{GSp}_{2n}^+(\mathbb{R}) = \{ \gamma \in \text{GSp}_{2n}^+(\mathbb{R}) : \mu_n(\gamma) > 0 \}$ action on $\mathfrak{h}^n$ is given by

$$
\gamma Z = (a_\gamma Z + b_\gamma)(c_\gamma Z + d_\gamma)^{-1}
$$

for $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \text{GSp}_{2n}^+(\mathbb{R})$ and $Z \in \mathfrak{h}^n$. 

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For $M \geq 1$, the congruence subgroup of $\operatorname{Sp}_{2n}(\mathbb{Z})$ is defined as:

$$
\Gamma_0^{(n)}(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{Sp}_{2n}(\mathbb{Z}) : c \equiv 0 \pmod{M} \right\}.
$$

Note this is the natural generalization of $\Gamma_0(M) \subset \operatorname{SL}_{2}(\mathbb{Z})$ to this setting.
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**Definition**

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Note this is the natural generalization of $\Gamma_0(M) \subset \text{SL}_2(\mathbb{Z})$ to this setting.
Action of $\text{GSp}^+_2(\mathbb{R})$ on functions $F : \mathfrak{h}^n \to \mathbb{C}$

For $\gamma \in \text{GSp}^+_2(\mathbb{R})$ and $Z \in \mathfrak{h}^n$,

$$j(\gamma, Z) = \det(c_{\gamma}Z + d_{\gamma}).$$

Let $\kappa$ be a positive integer. Given a function $F : \mathfrak{h}^n \to \mathbb{C}$, we set

$$(F|_{\kappa\gamma})(Z) = \mu_n(\gamma)^{n\kappa/2}j(\gamma, Z)^{-\kappa}F(\gamma Z).$$
Action of \( \text{GSp}_{2n}^+(\mathbb{R}) \) on functions \( F : \mathfrak{h}^n \to \mathbb{C} \)

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\[
(F|_{\kappa \gamma})(Z) = \mu_n(\gamma)^{\kappa/2} j(\gamma, Z)^{-\kappa} F(\gamma Z).
\]

**Definition**

We say such an \( F \) is a Siegel modular form of degree \( n \), weight \( \kappa \), and level \( \Gamma \) if \( F \) is a holomorphic function and satisfies

\[
(F|_{\kappa \gamma})(Z) = F(Z)
\]

for all \( \gamma \in \Gamma \).

The space of Siegel modular forms of weight \( \kappa \) and level \( \Gamma \) is \( M_\kappa(\Gamma) \).
If $F$ is a Siegel modular form of degree $n > 1$, it has a Fourier expansion of the form

$$F(Z) = \sum_{T \in S_n^{\geq 0}(\mathbb{Z})} a_F(T)e(\operatorname{Tr}(TZ))$$

where $S_n^{\geq 0}(\mathbb{Z})$ is the semi-group of $nxn$ positive semi-definite half-integral symmetric matrices.
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where $S_{n}^{\geq 0}(\mathbb{Z})$ is the semi-group of $n \times n$ positive semi-definite half-integral symmetric matrices.

**Definition**

$F$ is a **Siegel cusp form** ($F \in S_{\kappa}(\Gamma)$) if and only if $a_F(T) = 0$ when $\det T = 0$ and

$$F(Z) = \sum_{T \in S_{n}^{\geq 0}(\mathbb{Z}), T > 0} a_F(T)e(\text{Tr}(TZ)).$$
Consider the Siegel upper half plane $\mathfrak{h}^2$ of degree 2 and the following form

**Definition**

Let $t \geq 1$ be an integer and let

$$P_t = \text{diag}(1, t) = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$$

and we consider the skew-symmetric bilinear form written in block form

$$J_t = \begin{pmatrix} 0 & P_t \\ -P_t & 0 \end{pmatrix}$$
The Paramodular Group of level $t \geq 1$

**Definition**

The Paramodular Group of level $t$

$$\Gamma[t] = \text{Sp}_4(\mathbb{Q}) \cap \left\{ \begin{pmatrix} \mathbb{Z} & t\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & t^{-1}\mathbb{Z} \\ \mathbb{Z} & t\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ t\mathbb{Z} & t\mathbb{Z} & t\mathbb{Z} & \mathbb{Z} \end{pmatrix} \right\}$$
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Note that for $t = 1$ this group is the Siegel Modular group $\text{Sp}_4(\mathbb{Z})$. 

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\[ \mathfrak{h}^2 / \Gamma[t] \]

is the moduli space of abelian surfaces \( S \) with polarization of type \((1, t)\) (conductor \( t \)).
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is the moduli space of abelian surfaces \( S \) with polarization of type \((1, t)\) (conductor \( t \)).

We may write \( S \) as a two dimensional complex torus
\[ \mathbb{C}^2 / L \]
where
\[ L = \mathbb{Z}\mathbb{Z}^2 \oplus P_t\mathbb{Z}^2 \]
\( Z \in \mathbb{H}^2 \) and \( P_t = \text{diag}(1, t) \).

The polarization with respect to this basis is given by the form \( J_t \).
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1. Paramodular levels are the “right” levels to work with in terms of a newform theory. Roberts and Schmidt have a local newform theory for paramodular levels.
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2. The following conjecture of Brumer and Kramer:
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1. Paramodular levels are the “right” levels to work with in terms of a newform theory. Roberts and Schmidt have a local newform theory for paramodular levels.

2. The following conjecture of Brumer and Kramer:

**Conjecture**

There is a one-to-one correspondence between isogeny classes of abelian surfaces $A/\mathbb{Q}$ of conductor $t$ with $\text{End}_\mathbb{Q}(A) = \mathbb{Z}$ and weight 2 and level $\Gamma[t]$ newforms $F$ with rational eigenvalues, not in the span of Gritsenko lifts, such that $L(s, A) = L(s, F, \text{spin})$. 
The Jacobi group

Let $\Gamma$ be a congruence subgroup of $\text{SL}_2(\mathbb{Z})$. The Jacobi group $\Gamma^J := \Gamma \rtimes \mathbb{Z}^2$ is

$$\Gamma^J := \{ (M, X) : (M, X)(M', X') = (MM', XM' + X') \}$$

for all $M, M' \in \Gamma$ and $X = [\lambda, \mu], X' = [\alpha, \beta] \in \mathbb{Z}^2$. 
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If $\Gamma = \text{SL}_2(\mathbb{Z})$, $\text{SL}_2(\mathbb{Z})^J$ is called the full Jacobi group.
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If $\Gamma = SL_2(\mathbb{Z})$, $SL_2(\mathbb{Z})^J$ is called the the full Jacobi group.

The space of Jacobi cusp forms of weight $\kappa$, index $t$ and level $\Gamma_0(M)^J$ (resp $SL_2(\mathbb{Z})^J$) is $J^c_{\kappa, t}(\Gamma_0(M)^J)$ (resp. $J^c_{\kappa, t}$).
Let $f \in S_{2\kappa-2}^{\text{new}}(\Gamma_0(m))$ be a newform. There are essentially two classical ways to construct a Siegel modular form $F_f$ associated to $f$ (referred to as a Saito-Kurokawa lift of $f$):
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1. If $\kappa$ is even, one has a lifting $F_f \in S_{\kappa}(\Gamma_0^2(m))$ due to numerous people: Manickham-Ramakrishnan-Vasudevan, Piatetski-Shapiro, Schmidt, Skinner-Urban.
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2. If $f \in S_{2\kappa - 2}^{\text{new},-}(\Gamma_0(m))$, then one has a lifting $F_f \in S_{\kappa}(\Gamma[m])$ due to Skoruppa-Zagier and Gritsenko.
Schmidt has given a representation theoretic construction of each of the classical Saito-Kurokawa lifts assuming $m$ is odd and square-free.
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His construction is local in nature, so one can form many Saito-Kurokawa lifts from a single $f$. 
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His construction is local in nature, so one can form many Saito-Kurokawa lifts from a single $f$.

In this talk we will give a precise statement about “mixed-level” Saito-Kurokawa lifts and outline a classical construction of such liftings.
Definition

Let $t$ and $M \in \mathbb{N}$.

$$\Gamma_M[t] = \text{Sp}_4(\mathbb{Q}) \cap \left\{ \begin{pmatrix} \mathbb{Z} & t\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & t^{-1}\mathbb{Z} \\ M\mathbb{Z} & Mt\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ Mt\mathbb{Z} & Mt\mathbb{Z} & t\mathbb{Z} & \mathbb{Z} \end{pmatrix} \right\}$$
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1. The group $\Gamma_{M[t]}$ has mixed levels in it; it is of paramodular level $t$ and of congruence level $M$.
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For $M = 1$, $\Gamma_M[t] = \Gamma[t]$; hence we can say that the paramodular group $\Gamma[t]$ is of congruence level 1.
Definition

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Mt\mathbb{Z} & Mt\mathbb{Z} & t\mathbb{Z} & \mathbb{Z}
\end{pmatrix} \right\}$$

1. The group $\Gamma_M[t]$ has mixed levels in it; it is of paramodular level $t$ and of congruence level $M$.

2. For $M = 1$, $\Gamma_M[t] = \Gamma[t]$; hence we can say that the paramodular group $\Gamma[t]$ is of congruence level 1.

3. Similarly, for $t = 1$, the group $\Gamma_M[t]$ is the congruence subgroup $\Gamma_0^{(2)}(M)$ of $\text{Sp}_4(\mathbb{Z})$. 
Let $t, M \in \mathbb{N}$ be odd square-free such that $\gcd(M, t) = 1$. Fix $f$ a newform in $S_{2\kappa-2}^{\text{new}}(tM)$.

Depending on certain choices of a set $S$ of places $p$ with condition on the Atkin-Lehner eigenvalue at $p$, the possible Saito-Kurokawa lifts that $f$ can have are the following:

- If $f \in S_{2\kappa-2}^{\text{new, -}}(\Gamma_0(Mt))$ then $F_f \in S_{\kappa}(\Gamma[tM]).$
- If $\kappa$ is even
  1. $F_f \in S_{\kappa}(\Gamma_M[t]).$
  2. $F_f \in S_{\kappa}(\Gamma_t[M]).$
  3. $F_f \in S_{\kappa}(\Gamma_0^{(2)}(tM)).$
A couple definitions before the main theorem

Define

\[ S_{2\kappa-2}^t(\Gamma_0(Mt)) = \{ f \in S_{2\kappa-2}(\Gamma_0(Mt)) : f|_{W_t} = (-1)^\kappa f \}. \]
A couple definitions before the main theorem

Define

\[ S_{2\kappa-2}^t(\Gamma_0(Mt)) = \{ f \in S_{2\kappa-2}(\Gamma_0(Mt)) : f|_{W_t} = (-1)^\kappa f \}. \]

Note that in the case \( M = 1 \) this coincides with \( S_{2\kappa-2}(\Gamma_0(t)) \).
Define

\[ S^t_{2\kappa-2}(\Gamma_0(Mt)) = \{ f \in S_{2\kappa-2}(\Gamma_0(Mt)) : f|W_t = (-1)^\kappa f \}. \]

Note that in the case \( M = 1 \) this coincides with \( S^-_{2\kappa-2}(\Gamma_0(t)) \).

Recall that for \( F \in S_\kappa(\Gamma) \) a Siegel eigenform, the Spinor \( L \)-function is defined by

\[ L(s, F, \text{spin}) = \zeta(2s - 2k + 4) \sum_{n \geq 1} \lambda_F(n)n^{-s}. \]
(Brown, Z.) Let $M$ and $t$ be odd square-free integers, $\gcd(M, t) = 1$, $\kappa \geq 2$ an even integer, and $f \in S_{2\kappa-2}^{t,\text{new}}(\Gamma_0(Mt))$ a newform. Let $\epsilon_p$ be the eigenvalue of $f$ under the Atkin-Lehner involution at $p$ and let $\eta_p$ be the Atkin-Lehner involution of degree 2 at $p$. There exists an eigenform $F_f \in S_{\kappa}(\Gamma_M[t])$, unique up to constant multiples, whose Spinor $L$-function is given by

$$L(s, F_f, \text{spin}) = \left( \prod_{\substack{p \mid M \\ \epsilon_p = -1}} (1 - p^{-s+\kappa-1}) \right) \zeta(s-\kappa+1)\zeta(s-\kappa+2)L(s, f).$$

Moreover, for each $p \mid t$ we have $\eta_p F_f = \epsilon_p F_f$ and for each $p \mid M$ we have $\eta_p F_f = F_f$. 

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On Saito-Kurokawa Lifts
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2. We can construct the lifting without the requirement $M$ and $t$ be square-free. We only require square-free in order to get uniqueness (it won’t be unique in general) as well as to get the correct $L$-functions.
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We can construct the lifting without the requirement $M$ and $t$ be square-free. We only require square-free in order to get uniqueness (it won’t be unique in general) as well as to get the correct $L$-functions.

We use representation theoretic methods to get uniqueness and the result on the $L$-functions, which is why we require odd and square-free.
Outline of the proof

We construct $F_f$ via a series of liftings:

1. First we lift from $S_{	ext{new}}^\kappa - 2(\Gamma_0(M))$ to $S_{\text{Koh}}^\kappa - 1/2(\Gamma_0(4M))$ via the Shintani lifting.

2. Next we lift from $S_{\text{Koh}}^\kappa - 1/2(\Gamma_0(M))$ to $J_{c, t}(\Gamma_0(M))$.

3. Finally, we generalize Gritsenko's lifting to get a map $J_{c, t}(\Gamma_0(M))$ to $S^\kappa(\Gamma_M[t])$. This is the focus of the remainder of the talk.
We construct $F_f$ via a series of liftings:

1. First we lift from $S_{2\kappa-2}^{\text{new}}(\Gamma_0(Mt))$ to $S_{\kappa-1/2}^{\text{Koh}}(\Gamma_0(4Mt))$ via the Shintani lifting.

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Outline of the proof

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3. Finally, we generalize Gritsenko’s lifting to get a map $J_{\kappa,t}^c(\Gamma_0(M)^J)$ to $S_\kappa(\Gamma_M[t])$. This is the focus of the remainder of the talk.
Outline of the proof

We construct $F_f$ via a series of liftings:

1. First we lift from $S_{2\kappa-2}^{new}(\Gamma_0(Mt))$ to $S_{\kappa-1/2}^{Koh}(\Gamma_0(4Mt))$ via the Shintani lifting.

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3. Finally, we generalize Gritsenko’s lifting to get a map $J_{\kappa,t}^c(\Gamma_0(M)^J)$ to $S_{\kappa}(\Gamma_M[t])$. This is the focus of the remainder of the talk.
Fourier Series of a Siegel Modular Form of degree 2

For $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathfrak{h}^2$ we write it as a row vector $(\tau, z, \tau')$, $\tau, \tau' \in \mathfrak{h}^1$, $z \in \mathbb{C}$, and $\text{Im}(z)^2 < \text{Im}(\tau) \text{Im}(\tau')$. 

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For $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathfrak{h}^2$ we write it as a row vector $(\tau, z, \tau')$, $\tau, \tau' \in \mathfrak{h}^1$, $z \in \mathbb{C}$, and $\text{Im}(z)^2 < \text{Im}(\tau) \text{Im}(\tau')$.

We write $F(\tau, z, \tau')$ instead of $F(Z)$. 
Fourier Series of a Siegel Modular Form of degree 2

For $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathfrak{h}^2$ we write it as a row vector $(\tau, z, \tau')$, $\tau, \tau' \in \mathfrak{h}^1$, $z \in \mathbb{C}$, and $\text{Im}(z)^2 < \text{Im}(\tau) \text{Im}(\tau')$.

We write $F(\tau, z, \tau')$ instead of $F(Z)$.

For every $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} \in S_{2}^{>0}(\mathbb{Z})$, we write $a_F(n, r, m)$ for $a_F(T)$ where $n, r, m \in \mathbb{Z}$, $n, m \geq 0$ and $r^2 \leq 4mn$. 
Fourier Series of a Siegel Modular Form of degree 2

- For \( Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathfrak{h}^2 \), we write it as a row vector \((\tau, z, \tau')\), \(\tau, \tau' \in \mathfrak{h}^1, z \in \mathbb{C}\), and \(\text{Im}(z)^2 < \text{Im}(\tau) \text{Im}(\tau')\).

- We write \( F(\tau, z, \tau') \) instead of \( F(Z) \).

- For every \( T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} \in S_{2}^{>0}(\mathbb{Z}) \), we write \( a_F(n, r, m) \) for \( a_F(T) \) where \( n, r, m \in \mathbb{Z}, n, m \geq 0 \) and \( r^2 \leq 4mn \).

The Fourier expansion of \( F \in S_{\kappa}(\Gamma) \) takes the form

\[
F(\tau, z, \tau') = \sum_{m,n,r \in \mathbb{Z}, m,n,4mn-r^2>0} a_F(n, r, m)e(n\tau + rz + m\tau').
\]

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On Saito-Kurokawa Lifts
Fourier-Jacobi expansion of Mixed Congruence level Paramodular forms

Let $F \in M_\kappa(\Gamma_M[t])$. We can rewrite its Fourier expansion as

$$F(\tau, z, \tau') = \sum_{m \geq 0} \phi_{mt}(\tau, z) e(2\pi i (mt)\tau').$$
Let $F \in M_\kappa(\Gamma_M[t])$. We can rewrite its Fourier expansion as

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**Theorem (Z.)**

Let $F \in M_\kappa(\Gamma_M[t])$. For each $m$, its Fourier-Jacobi coefficient $\phi_{mt}$ belongs to $J_{\kappa,mt}(\Gamma_0(M)^J)$. 

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**Theorem (Z.)**

Let $F \in M_{\kappa}(\Gamma_M[t])$. For each $m$, its Fourier-Jacobi coefficient $\phi_{mt}$ belongs to $J_{\kappa,mt}(\Gamma_0(M)^J)$.

**Corollary**

Let $F \in M_{\kappa}(\Gamma[t])$. For each $m$, $F$’s Fourier-Jacobi coefficient $\phi_{mt}$ belongs to $J_{\kappa,mt}$. 
Theorem (Z.)

Let $\phi_t$ be a Jacobi cusp form of weight $\kappa \geq 2$, index $t$, and level $\Gamma_0(M)^J$ with Fourier expansion

$$\phi_t(\tau, z) = \sum_{n, r \in \mathbb{Z}, n \geq 0 \atop 4nt > r^2} c(n, r)e(2\pi i(n\tau + rz)).$$

Then

$$G_M(\phi_t)(\tau, z, \tau') := \sum_{m \geq 1} V_m(\phi_t)e(2\pi imt\tau')$$

lies in $S_\kappa(\Gamma_M[t])$ where $V_m$ is the index-shifting operator.
For $t = 1$ the lifting
\[ \mathcal{G}_M : \phi_1 \rightarrow \mathcal{G}_M(\phi_1) \]
is the Maass lifting with level $M$
\[ \mathcal{V} : J_{\kappa,1}^c(\Gamma_0(M)^J) \rightarrow S_{\kappa}(\Gamma_0^{(2)}(M)). \]
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2. For $M = 1$ the lifting
   \[ \mathcal{G}_M : \phi_t \rightarrow \mathcal{G}_M(\phi_t) \]
   is Gritsenko’s lifting
   \[ \mathcal{G} : J_{\kappa,t}^c \rightarrow S_\kappa(\Gamma[t]). \]
Let $F \in S_\kappa(\Gamma_M[t])$ have Fourier-Jacobi expansion

$$F(\tau, z, \tau') = \sum_{m \geq 1} \phi_{mt}(\tau, z)e(2\pi imt\tau').$$
Let $F \in S_\kappa(\Gamma_M[t])$ have Fourier-Jacobi expansion

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One obtains an infinite family of liftings given by

$$S_\kappa(\Gamma_M[t]) \twoheadrightarrow \prod_{m \geq 1} J_{\kappa, mt}(\Gamma_0(M)^J) \twoheadrightarrow \prod_{m \geq 1} S_\kappa(\Gamma_M[mt])$$

$$F \twoheadrightarrow (\phi_{mt})_{m \geq 1} \twoheadrightarrow (\mathcal{G}(\phi_{mt}))_{m \geq 1}.$$

One can iterate this process indefinitely.
Corollary (Z.)

The map $G_M : J_{\kappa,t}^c(\Gamma_0(M)^J) \to S_\kappa(\Gamma_M[t])$ is injective.
Corollary (Z.)

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The composition

$$J_{\kappa, t}(\Gamma_0(M)^J) \to S_{\kappa}(\Gamma M[t]) \to \prod_{m \in \mathbb{N}} J_{\kappa, mt}(\Gamma_0(M)^J) \to J_{\kappa, t}(\Gamma_0(M)^J)$$

is the identity.
The image of the lifting $\mathcal{G}_M$ is the subspace of $S_\kappa(\Gamma_M[t])$ consisting of modular forms whose Fourier coefficients satisfy the following relations

$$a(n, r, mt) = \sum_{d | (n, r, m)} d^{\kappa-1} c\left(\frac{nm}{d^2}, \frac{r}{d}\right).$$

Note that for $t = 1$ these relations are exactly the Maass relations satisfied by the classical Saito-Kurokawa lifting of level $\Gamma_M(2)$.

We denote this subspace by $S^{*}_\kappa(\Gamma_M[t])$ to make it consistent with the Maass subspace notation.

Corollary (Z.)

We have the following isomorphism of vector spaces

$$J_c\kappa, t(\Gamma_0(M)) \cong S^{*}_\kappa(\Gamma_M[t]).$$
The image of the lifting $G_M$ is the subspace of $S_κ(Γ_M[t])$ consisting of modular forms whose Fourier coefficients satisfy the following relations

$$a(n, r, mt) = \sum_{d \mid (n, r, m), \quad r^2 < 4nm, \quad (d, M) = 1} d^{κ-1} c\left(\frac{nm}{d^2}, \frac{r}{d}\right).$$

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$$a(n, r, mt) = \sum_{d \mid (n, r, m) \atop r^2 < 4nmt \atop (d, M) = 1} d^{\kappa-1} c\left(\frac{nm}{d^2}, \frac{r}{d}\right).$$

Note that for $t = 1$ these relations are exactly the Maass relations satisfied by the classical Saito-Kurokawa lifting of level $\Gamma_0^{(2)}(M)$. We denote this subspace by $S_{\kappa}^*(\Gamma_M[t])$ to make it consistent with the Maass subspace notation.
The image of the lifting $G_M$ is the subspace of $S_\kappa(\Gamma_M[t])$ consisting of modular forms whose Fourier coefficients satisfy the following relations

$$a(n, r, mt) = \sum_{d \mid (n, r, m), r^2 < 4nmt, (d, M) = 1} d^{\kappa - 1} c \left( \frac{nm}{d^2}, \frac{r}{d} \right).$$

Note that for $t = 1$ these relations are exactly the Maass relations satisfied by the classical Saito-Kurokawa lifting of level $\Gamma_0^{(2)}(M)$. We denote this subspace by $S^*_\kappa(\Gamma_M[t])$ to make it consistent with the Maass subspace notation.

**Corollary (Z.)**

We have the following isomorphism of vector spaces

$$J^c_{\kappa, t}(\Gamma_0(M)^J) \cong S^*_\kappa(\Gamma_M[t]).$$
Lemma (Z.)

The Mixed level lifting $G_M$ is Hecke equivariant with respect to the Hecke algebra homomorphism $\iota : \mathbb{T}_{\mathbb{Z}}^{S,tM} \rightarrow \mathbb{T}_{\mathbb{Z}}^{J,tM}$ given by

$$\iota(T_S(p)) = -T_J(p) + p^{k-1} + p^{k-2} \quad (p \nmid tM),$$

$$\iota(T'_S(p)) = (p^{k-1} + p^{k-2})T_J(p) + 2p^{2k-3} + p^{2k-4} \quad (p \nmid tM).$$

Equivalently, the lifting $G_M$ satisfies

$$(G_M(\phi)| T = G_M(\phi|\iota(T))$$

for any $T \in \mathbb{T}_{\mathbb{Z}}^{S,tM}$. 
Some future directions

1. Study the algebraicity of special values of the $L$-functions of the mixed level liftings. In the case $M = 1$ or $t = 1$ this is known by work of Brown-Pitale.
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2. Show that these liftings can be put in a Hida family. The case of $M = 1$ and $t = 1$ is known by Guerzhoy, Kawamura, the case of $t = 1$ is known by work of Brown-Klosin and the case $M = 1$ is known by work of Skinner-Urban.
Some future directions

1. Study the algebraicity of special values of the $L$-functions of the mixed level liftings. In the case $M = 1$ or $t = 1$ this is known by work of Brown-Pitale.

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3. Construct congruences between the lifted forms and non-lifted Siegel forms. This has applications to a further conjecture of Brumer-Kramer. Namely, if $M = 1$, they conjecture a congruence modulo $p$ between a Saito-Kurokawa lift and a non-lifted form should correspond to a $p$-torsion point on the abelian surface given by the non-lifted form. So far their congruences are all computational.