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Automorphic Descent method, first introduced by David Ginzburg, Stephen Rallis and David Soudry in 1998, is to construct certain cuspidal automorphic forms on classical groups in terms of these on general linear groups, which are related by the Langlands functoriality.
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Square-Integrable Automorphic Forms

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- For example,

\[ \text{SO}_m := \{ g \in \text{GL}_m \mid ^t g J_m g = J_m, \det g = 1 \}, \]

with $J_m$ defined inductively by $J_m := \begin{pmatrix} & 1 \\ J_{m-1} & \end{pmatrix}$. 


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Square-Integrable Automorphic Forms

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Constructions of Automorphic Forms
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- $L^2(X_G)$ denotes the space of functions: $\phi : X_G \to \mathbb{C}$ such that

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Theorem (Gelfand, Graev, Piatetski-Shapiro, Langlands) \[ L^2_d(X_G) = \bigoplus_{\pi \in \hat{G}(\mathbb{A})} m_d(\pi) \cdot V_\pi \] with the multiplicity $m_d(\pi)$ finite, where $\hat{G}(\mathbb{A})$ is the unitary dual of $G(\mathbb{A})$.

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$$m_d(\pi) \leq \begin{cases} 
1, & \text{if } G = \text{SO}_{2n+1}, \text{Sp}_{2n} \\
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- $G = \text{GL}_n$, $m_d(\pi) \leq 1$ (Shalika; Piatetski-Shapiro; Moeglin-Waldspurger);
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- $G = \text{SL}_n (n \geq 3)$, $m_d(\pi) > 1$ for some $\pi$ (Blasius; Lapid);
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How to Construct Automorphic Forms?

- This is an easy-hard problem in general.

Let $\Theta(g, h)$ be an automorphic function on $G(A) \times H(A)$ and $\phi, \varphi$ be automorphic forms on $G(A), H(A)$, respectively. Consider the following integral (assuming convergence)

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\int_{G(A) \times H(A)} \Theta(g, h) \phi(g) \varphi(h) \, dg \, dh.
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If (1) is nonzero, then the integration along $dh$ will construct automorphic functions on $G(A)$ by means of those on $H(A)$, while the integration along $dg$ will produce the opposite direction construction. Hence the construction is easy!

However, if $\phi$ and $\varphi$ need to satisfy a particular relation, say, Langlands functoriality, for instance, it is in general a very hard problem to design the kernel function $\Theta(g, h)$!
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- Automorphic representations are $G(\mathbb{A})$-submodules in $L^2(\mathcal{X}_G)$. The set of all irreducible ones is denoted by $\Pi^a(G)$.
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  where $c(\pi_v)$ is a conjugacy class of semi-simple elements in the Langlands dual group $L^G$.
- $L^G = G^\vee(\mathbb{C}) \rtimes \Gamma_\mathbb{Q}$, where $G^\vee(\mathbb{C})$ is given by
  \[ G \iff (X, \Delta; X^\vee, \Delta^\vee) \]
  \[ G^\vee(\mathbb{C}) \iff (X^\vee, \Delta^\vee; X, \Delta) \]
- $GL_n^\vee(\mathbb{C}) = GL_n(\mathbb{C})$ and $SO_{2n+1}^\vee(\mathbb{C}) = Sp_{2n}(\mathbb{C})$. 
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- $S$ denotes any finite set of primes $p$ and $\infty$.
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Constructions of Automorphic Forms
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Problems:

1. The image $c(\Pi^a(G))$ in $\mathcal{C}(G)$ (Ramanujan Conjecture).
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Langlands Functoriality

Langlands Functoriality Conjecture consists of Two Parts:

▶ **Transfer:** $G, H$ reductive algebraic $\mathbb{Q}$-groups and a group homomorphism

$$\rho : \mathbb{L}H \rightarrow \mathbb{L}G,$$

which is compatible with the action of $\Gamma_{\mathbb{Q}}$. For any $\sigma \in \Pi^a(H)$, $\exists$ a $\pi \in \Pi^a(G)$ s.t.

$$c(\rho^a(\sigma)) = c(\pi)$$

as conjugacy classes in $\mathbb{L}G$, where $c(\rho^a(\sigma)) = \{\rho(c(\sigma_v))\}$. 

▶ **Thickness:** For each tempered $\pi \in \Pi^a(G)$, $\exists$ an $H$; a thick $\sigma \in \Pi^a(H)$, s.t. $\pi$ is a Langlands functorial transfer of $\sigma$.

The thickness of $\sigma$ is defined in terms of invariant theory of $\mathbb{L}H$ and analytic properties of automorphic $\mathbb{L}$-functions attached to $\sigma$, and was first introduced by Langlands in his Shaw prize lecture (2007, Shahidi’s volume 2011).
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Example: Theta Correspondence

- \( O_{2m} \times \text{Sp}_{2n} \to \text{Sp}_{4mn} \), via the tensor product, forms a reductive dual pair in \( \text{Sp}_{4mn} \) in the sense of R. Howe.
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This leads to an explicit construction of the classical Shimura correspondence, which was the starting point of the classical theory of modular forms of half-integral weight.
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The theta correspondence may be formulated as

\[ \vdots \quad \vdots \]

\[ \text{Sp}_6 \quad \Pi^a(\text{Sp}_6) \]

\[ \uparrow \quad \uparrow \]

\[ \Pi^a(\text{SO}_{2m}) \quad \text{SO}_{2m} \quad \rightarrow \quad \text{Sp}_4 \quad \Pi^a(\text{Sp}_4) \]

\[ \downarrow \quad \uparrow \]

\[ \text{Sp}_2 \quad \Pi^a(\text{Sp}_2) \]

**Questions:** What is the structure of the first occurrence?
Example: Theta Correspondence–Properties

- J.-P. Waldspurger, 1980, the representation-theoretic approach to investigate the **Shimura correspondence**.

- S. Rallis, in 1982, the relation of the local Satake parameters of $\pi$ and $\sigma$ in terms of the Langlands functoriality.

- S. Rallis, in 1984, the **Tower Properties**. That is, the first occurrence is always cuspidal, and after that the theta correspondences are always nonzero, but noncuspidal.

- S. Kudla, in 1986, the local version of the **Tower Properties**.

- J. Adams, in 1989, formulated a conjecture (over $\mathbb{R}$) claiming that if (2) is nonzero for $(\pi, \sigma)$, then $\pi$ and $\sigma$ are related in terms of Arthur transfer, instead of the Langlands transfer.

- C. Moeglin, in 2011, discussed the relation of theta correspondence, Adams’s Conjecture, and Arthur’s Conjecture on the discrete spectrum of automorphic forms.
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More Examples: Extended Theta Correspondences

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$$\int_{[G] \times [H]} \Theta(g,h)\phi(g)\varphi(h)dgdh$$  \hspace{1cm} (3)

where $\phi \in A(G)$ and $\varphi \in A(H)$, and $(G, H)$ forms a commuting pair in $G$. 

The names contributed to both local and global theories of the extended theta correspondences are: Kazhdan, Savin, Rubenthaler, Ginzburg, Rallis, Soudry, J.-S. Li, Gross, Jiang, Gan, Gurevich, and others.

Important applications were obtained, including the work of Gross and Savin on the existence of motives whose Galois group is the exceptional group of type $G_2$. 

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Constructions of Automorphic Forms
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Conjecture of Ginzburg-J.-Soudry (IMRN2011):
If integral (4) is nonzero for some $\varphi_\tau$ and $\varphi_\pi$, then the representation $\tau$ is the automorphic induction from $O_{2,\omega}(A)$, and further $\pi$ is the tensor product transfer from $\text{GL}_2 \times \text{GL}_n$.

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The above work for all classical groups (CKPSS(2004) and the book of Ginzburg-Rallis (2011)).
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$$
\begin{align*}
&\tau \quad GL_{2n} \\
&\downarrow FC \\
&LFT \\
&RES \\
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With the recent progress on the Fundamental Lemma and its variants by Ngo (and Waldspurger, Laumon-Ngo, ...), the **stable trace formula** of Arthur is able to prove the following key theorem, which was announced in Arthur’s 2005 Clay lecture notes and forthcoming book 2011.

**Theorem of Arthur:**

Let $G$ be a symplectic or orthogonal group. Then

\[ L^2(X_G) = \bigoplus_{\psi \in \Psi^2(G)} m_{\psi}( \bigoplus_{\pi \in \Pi(\psi)} m_{\pi} V_{\pi}) \]

with the multiplicity $m_{\pi}$ is 1 or 2.

$\Psi^2(G)$ is the set of global Arthur parameters of $G$.

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$$\Psi_2(G)$$

$\psi$\hline

ATF $\hookleftarrow$ \hline

LES $\overrightarrow{}$

$L^2_d(X_G) \ \Pi(\psi) \quad \rightarrow \quad E(\psi) \ \Pi^a(GL)$
Arthur’s Theorem and Langlands Functoriality

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\[
\Psi_2(G) \\
\psi \\
\downarrow \\
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E(\psi) \quad \Pi^a(\text{GL})
\]

- $\psi \mapsto \Pi(\psi)$ is given by Arthur stable trace formula.
- $\psi \mapsto E(\psi)$ is given by Langlands theory of Eisenstein series.
- $\Pi(\psi) \mapsto E(\psi)$ gives the existence of the Langlands transfer.
Two Problems Remain

Problems:

▶ (A) Refine the weak transfer $\Pi(\psi) \mapsto E(\psi)$ from classical group $G$ to general linear group $GL$ to the Langlands functorial transfer at all local places.

Possible Approach:

▶ (A) This is a deep arithmetic problem. At least, one needs the full theory of certain local $L$-functions and $\gamma$-factors. Also there is a serious problem with $E(\psi)$.
Two Problems Remain

Problems:

▶ (A) Refine the weak transfer $\Pi(\psi) \mapsto E(\psi)$ from classical group $G$ to general linear group $GL$ to the Langlands functorial transfer at all local places.

▶ (B) Construct explicitly members in the Arthur packet $\Pi(\psi)$.

Possible Approach:

▶ (A) This is a deep arithmetic problem. At least, one needs the full theory of certain local $L$-functions and $\gamma$-factors. Also there is a serious problem with $E(\psi)$.

▶ (B) We discuss recent progress of Ginzburg-J.-Soudry on construction of members in $\Pi(\psi)$, which is a generalization and combination of the theta liftings and the automorphic descents introduced by Ginzburg-Rallis-Soudry in 1998.
Global Arthur Parameters: $\Psi_2(G)$

- Take $G = \text{SO}_{2n+1}$, then $G^\vee(\mathbb{C}) = \text{Sp}_{2n}(\mathbb{C})$. 

Global Arthur Parameters: $Ψ_2(G)$

- Take $G = SO_{2n+1}$, then $G^\vee(\mathbb{C}) = Sp_{2n}(\mathbb{C})$.
- Each $\psi \in Ψ_2(SO_{2n+1})$ (Arthur parameters) is written as a formal sum of stable Arthur parameters:

$$\psi = ψ_1 ⊙ ψ_2 ⊙ \cdots ⊙ ψ_r$$

where $ψ_i = (τ_i, b_i)$, with $τ_i \in Π_{u,c,a}^{u}(GL_{a_i})$ and $a_i, b_i ≥ 1$. 
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- If $i \neq j$, either $\tau_i \not\sim \tau_j$ or $b_i \neq b_j$.
- $a_i$ and $b_j$ have to be in a certain parity, so that $a_i \cdot b_i$ is even and $\psi_i \in \Psi_2(\text{SO}_{a_i \cdot b_i+1})$. 

Conjectural Endoscopy Structure:

$$
\sum_{i=1}^{r} a_i \cdot b_i = \text{SO}_{\sum_{i=1}^{r} a_i \cdot b_i+1} \prod \Pi(\psi_1) \otimes \cdots \otimes \Pi(\psi_r)
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For other classical groups, the description of $\psi$ is similar.
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$$\text{SO}_{a_1 \cdot b_1+1} \times \cdots \times \text{SO}_{a_r \cdot b_r+1} \implies \text{SO}_{2n+1}$$

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Simple Arthur Parameter $\psi = (\tau, b)$

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- $\psi = (\tau, b)$ is self-dual iff $\tau$ is self-dual. Hence $\tau$ is either of symplectic type ($L^S(1, \tau, \wedge^2) = \infty$) or of orthogonal type ($L^S(1, \tau, \vee^2) = \infty$).
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If $\tau$ is of symplectic type ($a$ is even), then

$$(\tau, b) \text{ is } \begin{cases} \text{of symplectic type} & \text{if } b = 2l + 1; \\ \text{of orthogonal type} & \text{if } b = 2l. \end{cases}$$
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\]

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\[
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Simple $\psi = (\tau, b)$-Tower with $\tau$ symplectic

$$
\begin{align*}
&\uparrow \\
(\tau, 2m - 1) &\quad SO_{4nm-2n+1} \\
&\uparrow \\
&\quad SO_{4nm-4n} (\tau, 2m - 2) \\
&\uparrow \\
(\tau, 3) &\quad SO_{6n+1} \\
&\uparrow \\
&\quad SO_{4n} (\tau, 2) \\
(\tau, 1) &\quad SO_{2n+1}
\end{align*}
$$

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Simple $\psi = (\tau, b)$-Tower with $\tau$ symplectic, $L(\frac{1}{2}, \tau) \neq 0$
The first floor of the $\psi = (\tau, b)$-Tower is

\[
\begin{array}{ccc}
SO_{4n} & \overset{tc}{\leftrightarrow} & Sp_{4n} \\
\overset{lq}{\leftarrow} & \ & \overset{lq}{\rightarrow} \\
gg \downarrow & GL_{2n} & \downarrow fj \\
\overset{lt}{\rightarrow} & \ & \overset{lt}{\leftarrow} \\
SO_{2n+1} & \overset{tc}{\leftrightarrow} & \tilde{Sp}_{2n}
\end{array}
\]

For the p-adic case, see Ginzburg-Rallis-Soudry (1999), J.-Soudry (2003), and J.-Nien-Qin (2010).
\( \psi = (\tau, b) \)-Tower with \( \tau \) symplectic, \( \mathbb{L}(\frac{1}{2}, \tau) \neq 0 \)

The \( m = (2l - 1) \)-th floor of the \( \psi = (\tau, b) \)-Theta Tower is

\[
\begin{align*}
SO_{2nm+2n} & \leftrightarrow_{\text{tc}} & Sp_{2nm+2n} \\
& \leftarrow lq & \rightarrow lq \\
gg & \downarrow & GL_{2nm} & \downarrow fj \\
& \uparrow lt & \downarrow lt \\
SO_{2nm+1} & \leftrightarrow_{\text{tc}} & \tilde{Sp}_{2nm}
\end{align*}
\]
Consider the simple parameter $\psi = (\tau, b)$ with $\tau$ symplectic and $L(\frac{1}{2}, \tau) \neq 0$, and the following two basic triangles:

\begin{align*}
(\tau, 2m + 1) &\xrightarrow{\text{Sp}_{4nm+4n}} (\tau, 2m + 2) \\
&\downarrow \quad \uparrow \\
&\downarrow \quad \uparrow \\
&\text{Sp}_{4nm} \quad (\tau, 2m) \\
&\downarrow \quad \uparrow \\
&\text{Sp}_{4nm} \quad (\tau, 2m - 1)
\end{align*}

and

\begin{align*}
(\tau, 2m + 1) &\xrightarrow{\text{Sp}_{4nm+2n}} \\
&\uparrow \\
&\downarrow \\
&\text{Sp}_{4nm} \quad (\tau, 2m) \\
&\downarrow \\
&\text{Sp}_{4nm} \quad (\tau, 2m - 1)
\end{align*}
Basic Triangles in a $\psi = (\tau, b)$-Tower

Define $G_b(\mathbb{A})$ by

$$G_b(\mathbb{A}) = \begin{cases} \widetilde{Sp}_{4nm+2n}(\mathbb{A}) & \text{if } b = 2m + 1; \\ Sp_{4nm}(\mathbb{A}) & \text{if } b = 2m. \end{cases}$$

Then $G_b(\mathbb{A})$ has a standard parabolic subgroup

$$P_1^n = (GL_1^\times)^n \times G_{b-1})U_1^n.$$
Basic Triangles in a $\psi = (\tau, b)$-Tower

- Define $G_b(\mathbb{A})$ by

$$G_b(\mathbb{A}) = \begin{cases} \widetilde{Sp}_{4nm+2n}(\mathbb{A}) & \text{if } b = 2m + 1; \\ Sp_{4nm}(\mathbb{A}) & \text{if } b = 2m. \end{cases}$$

Then $G_b(\mathbb{A})$ has a standard parabolic subgroup

$$P_{1^n} = (GL_1^\times n \times G_{b-1}) U_{1^n}.$$  

- Consider the Fourier-Jacobi coefficient along the unipotent radical of $P_{1^n}$: $\varphi_b$ automorphic form on $G_b(\mathbb{A})$,

$$FJ_{b-1}^b(\varphi_b, \psi)(h) := \int_{U_{1^n}(\mathbb{Q}) \backslash U_{1^n}(\mathbb{A})} \varphi_b(uh) \tilde{\theta}^\psi(uh) \psi U_{1^n}(u) du.$$  

It is an automorphic form on $G_{b-1}(\mathbb{A})$. 

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Constructions of Automorphic Forms
Basic Triangles in a $\psi = (\tau, b)$-Tower

- Take $\pi_b \in \Pi^a(G_b)$. 

- $\text{Res}$ is to take certain residue of Eisenstein series attached to the datum $\tau | \cdot |_s \otimes \pi_b$. 

In general, this triangle is NOT commutative, and has no meaning related to Functoriality.
Basic Triangles in a $\psi = (\tau, b)$-Tower

- Take $\pi_b \in \Pi^a(G_b)$.
- Denote by $\mathcal{D}_{b-1, \psi}^b(\pi_b)$ the space generated by all Fourier-Jacobi coefficients $FJ_{b-1}^b(\varphi_b, \psi)$ with all $\varphi_b \in \pi_b$. 

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Basic Triangles in a $\psi = (\tau, b)$-Tower

- Take $\pi_b \in \Pi^a(G_b)$.
- Denote by $\mathcal{D}^b_{b-1,\psi}(\pi_b)$ the space generated by all Fourier-Jacobi coefficients $FJ^b_{b-1}(\varphi_b, \psi)$ with all $\varphi_b \in \pi_b$.
- We obtain

\[
\begin{array}{c}
\xymatrix{
G_{b+2}(\mathbb{A}) & D_{b+2,\psi} \\
G_{b+1}(\mathbb{A}) \\
G_b(\mathbb{A})}
\end{array}
\]

$\text{Res}$ is to take certain residue of Eisenstein series attached to the datum $\tau |\cdot|_s \otimes \pi_b$.

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Basic Triangles in a $\psi = (\tau, b)$-Tower

- Take $\pi_b \in \Pi^a(G_b)$.
- Denote by $\mathcal{D}^b_{b-1,\psi}(\pi_b)$ the space generated by all Fourier-Jacobi coefficients $FJ^b_{b-1}(\varphi_b, \psi)$ with all $\varphi_b \in \pi_b$.
- We obtain

\[ \begin{align*}
\mathcal{D}^{b+2}_{b+1,\psi} & \hookleftarrow G_{b+2}(\mathbb{A}) \\
G_{b+1}(\mathbb{A}) & \uparrow \text{Res} \\
\mathcal{D}^{b+1}_{b,\psi-1} & \hookrightarrow G_b(\mathbb{A})
\end{align*} \]

- $\text{Res}$ is to take certain residue of Eisenstein series attached to the datum $\tau| \cdot |^s \otimes \pi_b$. 

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Constructions of Automorphic Forms
Basic Triangles in a $\psi = (\tau, b)$-Tower

- Take $\pi_b \in \Pi^a(G_b)$.
- Denote by $D_{b-1,\psi}(\pi_b)$ the space generated by all Fourier-Jacobi coefficients $FJ_{b-1}(\varphi_b, \psi)$ with all $\varphi_b \in \pi_b$.
- We obtain

\[
\begin{array}{c}
G_{b+2}(\mathbb{A}) \\
D_{b+2}^{b+1, \psi} \\
G_{b+1}(\mathbb{A}) \\
D_{b, \psi-1}^{b+1} \\
G_b(\mathbb{A})
\end{array}
\]

- $\text{Res}$ is to take certain residue of Eisenstein series attached to the datum $\tau| \cdot |^s \otimes \pi_b$.
- In general, this triangle is NOT commutative, and has no meaning related to Functoriality.
Theorem (Ginzburg-J.-Soudry(2011))

Let $\pi_b$ be the residual representation of $G_b(\mathbb{A})$ with Arthur parameter $(\tau, b)$, $\tau$ symplectic and $L(\frac{1}{2}, \tau) \neq 0$. Then $\text{Res}(\pi_b)$ is the residual representation of $G_{b+2}(\mathbb{A})$ with Arthur parameter $(\tau, b + 2)$, and $D_{b+1, \psi}(\text{Res}(\pi_b))$ is the residual representation of $G_{b+1}(\mathbb{A})$ with Arthur parameter $(\tau, b + 1)$. Moreover, the basic triangle is a commutative diagram:

$$
\begin{array}{ccc}
G_{b+2}(\mathbb{A}) & (\tau, b + 2) \\
\downarrow \text{Res} & & \\
G_{b+1}(\mathbb{A}) & (\tau, b + 1) \\
\downarrow D_{b+1, \psi} & & \\
G_{b}(\mathbb{A}) & (\tau, b) \\
\end{array}
$$
Basic Triangles in a $\psi = (\tau, b)$-Tower

When $b = 1$, we have the following triangle

$$(\tau, 3) \xrightarrow{\text{Sp}_{6n}} \text{Sp}_{4n} (\tau, 2) \xleftarrow{\text{N}_{\text{Sp}_{4n}}(\tau, \psi)} (\tau, 1) \xleftarrow{\text{Sp}_{2n}} \text{N}_{\text{Sp}_{2n}}(\tau, \psi)$$
Basic Triangles in a $\psi = (\tau, b)$-Tower

- When $b = 1$, we have the following triangle

$$
\begin{align*}
(\tau, 3) & \xrightarrow{\widetilde{Sp}_{6n}} \leftarrow \uparrow \leftarrow \widetilde{Sp}_{4n} (\tau, 2) \xrightarrow{\mathcal{N}_{Sp_{4n}}(\tau, \psi)} \\
\mathcal{N}_{\widetilde{Sp}_{2n}}(\tau, \psi) & \leftarrow (\tau, 1) \xleftarrow{\widetilde{Sp}_{2n}}
\end{align*}
$$

- $\mathcal{N}_{\widetilde{Sp}_{2n}}(\tau, \psi)$ is the set of all irreducible, genuine, cuspidal automorphic representations of $\widetilde{Sp}_{2n}(\mathbb{A})$, which have $\tau$ as the $\psi$-weak Langlands transfer to $GL_{2n}(\mathbb{A})$. 
Basic Triangles in a $\psi = (\tau, b)$-Tower

- When $b = 1$, we have the following triangle

\[
\begin{array}{cccc}
(\tau, 3) & \tilde{\text{Sp}}_{6n} & \downarrow & (\tau, 2) \\
& & \uparrow & \mathcal{N}_{\text{Sp}_{4n}}(\tau, \psi) \\
\mathcal{N}_{\tilde{\text{Sp}}_{2n}}(\tau, \psi) & (\tau, 1) & \tilde{\text{Sp}}_{2n} & \mathcal{N}_{\text{Sp}_{4n}}(\tau, \psi)
\end{array}
\]

- $\mathcal{N}_{\tilde{\text{Sp}}_{2n}}(\tau, \psi)$ is the set of all irreducible, genuine, cuspidal automorphic representations of $\tilde{\text{Sp}}_{2n}(\mathbb{A})$, which have $\tau$ as the $\psi$-weak Langlands transfer to $\text{GL}_{2n}(\mathbb{A})$.

- $\mathcal{N}_{\text{Sp}_{4n}}(\tau, \psi)$ is the set of all irreducible automorphic representations $\pi$ of $\text{Sp}_{4n}(\mathbb{A})$, which occur in the discrete spectrum of $\text{Sp}_{4n}(\mathbb{A})$, have the Arthur parameter $(\tau, 2)$, and with nonzero Fourier-Jacobi $FJ^2_1(\varphi_\pi, \psi)$. 
Basic Triangles in a $\psi = (\tau, b)$-Tower

Theorem of Ginzburg-J.-Soudry (2011): Put \( \Phi := D_{2,\psi^{-1}} \circ \text{Res} \) and \( \Psi := D_{1,\psi} \). Under a mild assumption, \( \Phi \) and \( \Psi \) are bijections between \( \mathcal{N}_{\widetilde{\text{Sp}}_{2n}}(\tau, \psi) \) and \( \mathcal{N}_{\text{Sp}_{4n}}(\tau, \psi) \).
Theorem of Ginzburg-J.-Soudry (2011): Put $\Phi := D_{2,\psi}^3 \circ \text{Res}$ and $\Psi := D_{1,\psi}^2$. Under a mild assumption, $\Phi$ and $\Psi$ are bijections between $\widetilde{\mathcal{N}}_{\text{Sp}_{2n}}(\tau, \psi)$ and $\mathcal{N}_{\text{Sp}_{4n}}(\tau, \psi)$.

It is the first result which gives one-to-one relation between the set $\widetilde{\mathcal{N}}_{\text{Sp}_{2n}}(\tau, \psi)$ of tempered cuspidal automorphic representations (assuming the generalized Ramanujan conjecture) and the set $\mathcal{N}_{\text{Sp}_{4n}}(\tau, \psi)$ of non-tempered automorphic representations.
\[ \Phi := D_{2,\psi^{-1}}^3 \circ \text{Res} \quad \text{and} \quad \Psi := D_{1,\psi}^2. \]
Under a mild assumption, \( \Phi \) and \( \Psi \) are bijections between \( \mathcal{N}_{\tilde{\text{Sp}}_{2n}}(\tau, \psi) \) and \( \mathcal{N}_{\text{Sp}_{4n}}(\tau, \psi) \).

It is the first result which gives one-to-one relation between
the set \( \mathcal{N}_{\tilde{\text{Sp}}_{2n}}(\tau, \psi) \) of tempered cuspidal automorphic representations (assuming the generalized Ramanujan conjecture) and the set \( \mathcal{N}_{\text{Sp}_{4n}}(\tau, \psi) \) of non-tempered automorphic representations.

This is extension and refinement of the pioneer work of
Piatetski- Shapiro, of Maass-Zagier, and of Andrianov on the
Saito-Kurokawa conjecture (See also Ikeda 2006).
Theorem of Ginzburg-J.-Soudry (2011): Put $\Phi := D_{2,\psi}^3 \circ \text{Res}$ and $\Psi := D_{1,\psi}^2$. Under a mild assumption, $\Phi$ and $\Psi$ are bijections between $\mathcal{N}_{\tilde{\text{Sp}}_{2n}}(\tau, \psi)$ and $\mathcal{N}_{\text{Sp}_{4n}}(\tau, \psi)$.

It is the first result which gives one-to-one relation between the set $\mathcal{N}_{\tilde{\text{Sp}}_{2n}}(\tau, \psi)$ of tempered cuspidal automorphic representations (assuming the generalized Ramanujan conjecture) and the set $\mathcal{N}_{\text{Sp}_{4n}}(\tau, \psi)$ of non-tempered automorphic representations.

This is extension and refinement of the pioneer work of Piatetski-Shapiro, of Maass-Zagier, and of Andrianov on the Saito-Kurokawa conjecture (See also Ikeda 2006).

This, combining with a work of Ginzburg-Rallis-Soudry (2005), proves the generalization of Duke-Imamoglu-Ikeda lifting, which constructs a non-tempered cuspidal automorphic forms of $\text{Sp}_{2m}$ in terms of that of $\text{GL}_2$. 
The proof of the theorem uses the commutativity of the whole diagram:

\[ \begin{array}{c}
(\tau, 3) \xrightarrow{\tilde{S}p_{6n}} (\tau, 4) \\
\downarrow \\
(\tau, 2) \xleftarrow{\tilde{S}p_{4n}} \\
\uparrow \\
(\tau, 1) \xleftarrow{\tilde{S}p_{2n}} \\
\end{array} \]
The proof of the theorem uses the commutativity of the whole diagram:

\[
\text{Sp}_8 (\tau, 4) \quad \text{Sp}_4 (\tau, 2) \quad \text{Sp}_2 (\tau, 1) \quad \text{Sp}_6 (\tau, 3)
\]

It is highly nontrivial to show that for \( \varphi \in \mathcal{N}_{\text{Sp}_4} (\tau, \psi) \), the following

\[
\text{Res} \circ D^2_{1, \psi}(\varphi) = D^4_{3, \psi} \circ \text{Res}(\varphi)
\]

hold as residual automorphic forms for some choice of data.
Basic Triangles in a $\psi = (\tau, b)$-Tower

The proof of the theorem uses the commutativity of the whole diagram:

$$
\begin{array}{c}
\text{Sp}_{8n} \quad (\tau, 4) \\
\downarrow \\
(\tau, 3) \quad \tilde{\text{Sp}}_{6n} \\
\uparrow \\
\downarrow \\
\text{Sp}_{4n} \quad (\tau, 2) \\
\uparrow \\
(\tau, 1) \quad \tilde{\text{Sp}}_{2n}
\end{array}
$$

It is highly nontrivial to show that for $\varphi \in \mathcal{N}_{\text{Sp}_{4n}}(\tau, \psi)$, the following

$$
\text{Res} \circ D_{1,\psi}^2(\varphi) = D_{3,\psi}^4 \circ \text{Res}(\varphi)
$$

hold as residual automorphic forms for some choice of data.

The idea and the method are expected to work for $\psi = (\tau, b)$-towers of other classical groups.
Write a global Arthur parameter $\psi = (\tau, b) \boxplus \psi'$ with
- $(\tau, b) \in \Psi_2(\text{SO}_{2kb+1})$ ($\tau \in \mathcal{A}_u^c(\text{GL}_{2k})$, $b = 2m + 1$);
- $\psi' \in \Psi_2(\text{SO}_{2l+1})$. 

Look for an endoscopy structure $\text{SO}_{2kb+1} \times \text{SO}_{2l+1} \to \text{SO}_{2kb+2l+1}$.

Find an automorphic form $\theta_{\tau; k, b, l}$ on the product $\text{SO}_{2l+1}(A) \times \text{SO}_{2kb+2l+1}(A)$.

For $\varphi \in \mathcal{A}(\text{SO}_{2l+1})$, $\phi \in \mathcal{A}(\text{SO}_{2kb+2l+1})$, if
$$\int \theta_{\tau; k, b, l}(g, h) \varphi(g) \phi(h) dgdh \neq 0 \quad (6)$$
then $\varphi$ and $\phi$ are endoscopically related in terms of $\tau$.

$\theta_{\tau; k, b, l}$ is a Fourier coefficient $E_{\psi V \tau}$ of a residue $E_{\tau}$ of certain Eisenstein series on $\text{SO}_{2k}(2l+b+1)$, and the Fourier coefficient has stabilizer isomorphic to $\text{SO}_{2l+1} \times \text{SO}_{2l+2kb+1}$. 

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Constructions of Automorphic Forms
General Constructions of Ginzburg-J.-Soudry

- Write a global Arthur parameter \( \psi = (\tau, b) \boxtimes \psi' \) with
  - \((\tau, b) \in \Psi_2(SO_{2kb+1}) (\tau \in A^{u,c}(GL_{2k}), b = 2m + 1);\)
  - \(\psi' \in \Psi_2(SO_{2l+1}).\)

- Look for an endoscopy structure

\[
SO_{2kb+1} \times SO_{2l+1} \rightarrow SO_{2kb+2l+1}
\]
Write a global Arthur parameter $\psi = (\tau, b) \boxplus \psi'$ with
- $(\tau, b) \in \Psi_2(\text{SO}_{2kb+1})$ ($\tau \in A^{u,c}(\text{GL}_{2k})$, $b = 2m + 1$);
- $\psi' \in \Psi_2(\text{SO}_{2l+1})$.

Look for an endoscopy structure

\[ \text{SO}_{2kb+1} \times \text{SO}_{2l+1} \rightarrow \text{SO}_{2kb+2l+1} \]

Find an automorphic form $\theta_{\tau; k, b, l}$ on the product

\[ \text{SO}_{2l+1}(\mathbb{A}) \times \text{SO}_{2l+2kb+1}(\mathbb{A}). \]
Write a global Arthur parameter \( \psi = (\tau, b) \boxplus \psi' \) with
\begin{itemize}
  \item \((\tau, b) \in \Psi_2(\text{SO}_{2kb+1}) \quad (\tau \in A_u^\nu,(\text{GL}_{2k}), \ b = 2m + 1)\);
  \item \(\psi' \in \Psi_2(\text{SO}_{2l+1})\).
\end{itemize}
Look for an endoscopy structure
\[
\text{SO}_{2kb+1} \times \text{SO}_{2l+1} \rightarrow \text{SO}_{2kb+2l+1}
\]
Find an automorphic form \( \theta_{\tau;k,b,l} \) on the product
\[
\text{SO}_{2l+1}(\mathbb{A}) \times \text{SO}_{2l+2kb+1}(\mathbb{A}).
\]
For \( \phi \in A(\text{SO}_{2l+1}), \varphi \in A(\text{SO}_{2l+2kb+1}) \), if
\[
\int \theta_{\tau;k,b,l}(g, h)\phi(g)\overline{\varphi(h)}\,dg\,dh \neq 0 \tag{6}
\]
then \( \phi \) and \( \varphi \) are endoscopically related in terms of \( \tau \).
Write a global Arthur parameter $\psi = (\tau, b) \boxplus \psi'$ with
- $(\tau, b) \in \Psi_2(\text{SO}_{2kb+1})$ ($\tau \in A^{u,c}(\text{GL}_{2k}), \ b = 2m + 1$);
- $\psi' \in \Psi_2(\text{SO}_{2l+1})$.

Look for an endoscopy structure

$$\text{SO}_{2kb+1} \times \text{SO}_{2l+1} \rightarrow \text{SO}_{2kb+2l+1}$$

Find an automorphic form $\theta_{\tau;k,b,l}$ on the product

$$\text{SO}_{2l+1}(\mathbb{A}) \times \text{SO}_{2l+2kb+1}(\mathbb{A})$$

For $\phi \in A(\text{SO}_{2l+1}), \varphi \in A(\text{SO}_{2l+2kb+1}),$ if

$$\int \theta_{\tau;k,b,l}(g, h)\phi(g)\overline{\varphi(h)} \,dg \,dh \neq 0$$

then $\phi$ and $\varphi$ are endoscopically related in terms of $\tau$.

$\theta_{\tau;k,b,l}$ is a Fourier coefficient $E_{\tau}^{\psi \psi'}$ of a residue $E_{\tau}$ of certain Eisenstein series on $\text{SO}_{2k(2l+b+1)}$, and the Fourier coefficient has stabilizer isomorphic to $\text{SO}_{2l+1} \times \text{SO}_{2l+2kb+1}$. 

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Constructions of Automorphic Forms
The above construction is given by the following diagram.

\[
\begin{array}{ccc}
\text{SO}_{2k(2l+b+1)} & \stackrel{\uparrow}{\longrightarrow} & \text{GL}_{2k} \\
\downarrow & & \downarrow \theta_{\tau;k,b,l} \\
\text{SO}_{2kb+1} \times \text{SO}_{2l+1} & \overset{\leftrightarrow}{\longrightarrow} & \text{SO}_{2l+2kb+1} \\
\Pi((\tau, b)) & & \Pi(\psi') \\
\end{array}
\]

For general classical groups, such a construction can be formulated in a similar way.
The above construction is given by the following diagram.

\[
\begin{align*}
\text{SO}_{2k(2l+b+1)} & \hspace{1cm} \text{GL}_{2k} \hspace{1cm} \theta_{\tau;k,b,l} \\
\downarrow & \hspace{1cm} \downarrow \\
\text{SO}_{2kb+1} \times \text{SO}_{2l+1} & \hspace{1cm} \leftrightarrow \hspace{1cm} \text{SO}_{2l+2kb+1} \\
\Pi((\tau, b)) & \hspace{1cm} \Pi(\psi') & \hspace{1cm} \Pi(\psi)
\end{align*}
\]

- For general classical groups, such a construction can be formulated in a similar way.
When $\tau$ is symplectic, $b$ is odd; when $\tau$ is orthogonal, $b$ is even.

\[
\begin{align*}
\Pi(\psi_{SO_{2l+1}} \boxtimes (\tau, b + 2)) & \rightarrow \Pi(\psi_{SO_{2l+1}} \boxtimes (\tau, b)) \\
\Pi(\psi_{SO_{2l+1}}) & \rightarrow \Pi(\psi_{SO_{2l+1}}) \\
SO_{2l+1+2k(b+2)} & \rightarrow SO_{2l+1+2kb} \\
SO_{2l+1+2k(b-2)} & \rightarrow SO_{2l+1+2kb}
\end{align*}
\]
(GL\(_{2k}\), \(\tau\))-Tower (Orthogonal Type)

When \(\tau\) is symplectic, \(b\) is even; when \(\tau\) is orthogonal, \(b\) is odd.

\[
\begin{array}{c}
\vdots \\
\text{Sp}_{2l+2k(b+2)} & \Pi(\psi_{SO_{2l}} \boxtimes (\tau, b+2)) \\
\uparrow \\
\Pi(\psi_{SO_{2l}}) & \text{SO}_{2l} \\
\rightarrow & \text{Sp}_{2l+2kb} & \Pi(\psi_{SO_{2l}} \boxtimes (\tau, b)) \\
\downarrow & \uparrow \\
\text{Sp}_{2l+2k(b-2)} & \Pi(\psi_{SO_{2l}} \boxtimes (\tau, b-2)) \\
\vdots & \\
\end{array}
\]
The dual diagram can be formulated, just like the theta correspondences for reductive dual pairs, by using the metaplectic double cover \( \widetilde{Sp}_{2l} \).
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Expect the **first occurrence property** hold.
(GL_m, \tau)-Towers

- The dual diagram can be formulated, just like the theta correspondences for reductive dual pairs, by using the metaplectic double cover \( \widetilde{Sp}_{2l} \).
- Expect the **first occurrence property** hold.
- The compatibility of the \((GL_m, \tau)\)-towers with the **Arthur conjecture** generalizes the **Adams conjecture**, and the work of Moeglin as mentioned above.

- The local theory extends the **Howe duality principle**. Some work has been done through the work on the local descent by J.-Soudry (2003), J.-Nien-Qin (2010), and J.-Soudry (2011).
The dual diagram can be formulated, just like the theta correspondences for reductive dual pairs, by using the metaplectic double cover $\widetilde{Sp}_{2l}$.

Expect the first occurrence property hold.

The compatibility of the $(\text{GL}_m, \tau)$-towers with the Arthur conjecture generalizes the Adams conjecture, and the work of Moeglin as mentioned above.

Expect that $(\text{GL}_m, \tau)$-towers and the first occurrences are essentially related to the location of poles of the tensor product $L$-functions of classical group times a general linear group, as the theory of Kudla, Piatetski-Shapiro, and Rallis.
The dual diagram can be formulated, just like the theta correspondences for reductive dual pairs, by using the metaplectic double cover \( \widetilde{Sp}_{2l} \).

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Expect that \((\text{GL}_m, \tau)\)-towers and the first occurrences are essentially related to the location of poles of the tensor product \( L \)-functions of classical group times a general linear group, as the theory of Kudla, Piatetski-Shapiro, and Rallis.

The local theory extends the Howe duality principle. Some work has been done through the work on the local descent by J.-Soudry (2003), J.-Nien-Qin (2010), and J.-Soudry (2011).
A tempered Arthur parameter $\psi \in \Psi_2(G)$ has form

$$\psi = (\tau_1, 1) \boxplus \cdots \boxplus (\tau_r, 1) = (\tau_1, 1) \boxplus \psi_2.$$ 

Then there exists an endoscopy group $H_1 \times H_2$ of $G$, such that $\psi_1 = (\tau_1, 1) \in \Psi_2(H_1)$ and $\psi_2 \in \Psi_2(H_2)$.

Theorem (Ginzburg (2008) and GJS (in progress))

Let $\pi_1$ be a generic member in $\Pi(\psi_1)$, $\pi_2$ be a generic member in $\Pi(\psi_2)$, and $\pi$ be a generic member in $\Pi(\psi)$. 

A tempered Arthur parameter $\psi \in \Psi_2(G)$ has form

$$\psi = (\tau_1, 1) \boxplus \cdots \boxplus (\tau_r, 1) = (\tau_1, 1) \boxplus \psi_2.$$  

Then there exists an endoscopy group $H_1 \times H_2$ of $G$, such that $\psi_1 = (\tau_1, 1) \in \Psi_2(H_1)$ and $\psi_2 \in \Psi_2(H_2)$.

**Theorem (Ginzburg (2008) and GJS (in progress))**

Let $\pi_1$ be a generic member in $\Pi(\psi_1)$, $\pi_2$ be a generic member in $\Pi(\psi_2)$, and $\pi$ be a generic member in $\Pi(\psi)$.

- The constructed integral operator gives an endoscopy transfer from $H_1 \times H_2$ to $G$ taking $(\pi_1, \pi_2)$ to a generic member in $\Pi(\psi)$. 

A tempered Arthur parameter $\psi \in \Psi_2(G)$ has form

$$\psi = (\tau_1, 1) \boxplus \cdots \boxplus (\tau_r, 1) = (\tau_1, 1) \boxplus \psi_2.$$ 

Then there exists an endoscopy group $H_1 \times H_2$ of $G$, such that $\psi_1 = (\tau_1, 1) \in \Psi_2(H_1)$ and $\psi_2 \in \Psi_2(H_2)$.

**Theorem (Ginzburg (2008) and GJS (in progress))**

Let $\pi_1$ be a generic member in $\Pi(\psi_1)$, $\pi_2$ be a generic member in $\Pi(\psi_2)$, and $\pi$ be a generic member in $\Pi(\psi)$.

- The constructed integral operator gives an endoscopy transfer from $H_1 \times H_2$ to $G$ taking $(\pi_1, \pi_2)$ to a generic member in $\Pi(\psi)$.
- The integral operator determines a descent from $\pi$ to a generic member in $\Pi(\psi_2)$ by means of $\pi_1$. 
This theorem was first formulated by Ginzburg in 2008 Duke Math. J. and the odd orthogonal case was discussed there.

The general theorem including unitary groups was stated by Ginzburg-J.-Soudry (2010). The proof for the general classical groups involves some technical issues related to the certain properties in the relevant simple towers as discussed above and is our current work in progress.

Some special cases of non-tempered cases were also proved by Ginzburg-J.-Soudry (2011 preprints).

It remains mostly open for exceptional groups.

The explicit formula arose from such constructions has applications to other important problems, such as periods of automorphic forms, special values of $L$-functions, and so on.

It is very interesting to find such constructions for Langlands functorial transfers which are NOT of endoscopy type.
Remarks

- This theorem was first formulated by Ginzburg in 2008 Duke Math. J. and the odd orthogonal case was discussed there.
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- It remains mostly open for exceptional groups.
- The explicit formula arose from such constructions has applications to other important problems, such as periods of automorphic forms, special values of $L$-functions, and so on.
- It is very interesting to find such constructions for Langlands functorial transfers which are NOT of endoscopy type.
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