Modularity of Galois representations over imaginary quadratic fields

Krzysztof Klosin
(joint with T. Berger)

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Notation

- $F = \text{im. quadr. field}$, $p \nmid \# \text{Cl}_F d_K$, $\text{fix } p | p$
- $\Sigma =$ finite set of finite places of $F$, $p, \overline{p} \in \Sigma$, $G_\Sigma = \text{Gal}(F_\Sigma / F)$;
- $\mathcal{O} =$ ring of integers in a finite extension of $\mathbb{Q}_p$,
  $\varpi =$ uniformizer, $\mathcal{F} = \mathcal{O} / \varpi$;
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- $\wp = \text{uniformizer, } F = \mathcal{O}/\wp$
- $\Psi = (\text{unramified}) \text{ Hecke character of } \infty\text{-type } \overline{\mathbb{Z}}$, $\Psi_p : G_\Sigma \to \mathcal{O}^\times$ the associated Galois character, $\chi_0 = \Psi_p \pmod{\wp}$
Main Theorem

Theorem (Berger-K., 2011)

Suppose that $\dim_F \Ext^1_{F[G_{\Sigma}]}(\chi_0, 1) = 1$. Let $\rho: G_{\Sigma} \to \GL_2(\mathbb{Q}_p)$ be continuous and irreducible. Suppose that:

- $\det \rho = \Psi_p \rho_{ss} = 1 \oplus \chi_0$,
- $\rho$ is crystalline (or ordinary if $p$ splits),
- $\rho$ is minimally ramified.

Then $\rho$ is modular, i.e., $L(\rho \otimes \gamma, s) = L(s, \pi)$ for some automorphic representation $\pi$ of $\GL_2(A_F)$. 

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This is similar to a result of Skinner and Wiles which applies to $\mathbb{Q}$ or a totally real field, but their method fails for $F=$imaginary quadratic. An important step in their method is the existence of an ordinary, minimal deformation

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**Theorem (Berger-K.)**

*No such deformation $\rho$ exists for $F$.***
Remarks to the Main Theorem

- We do not follow [SW]-strategy. Instead we develop a commutative algebra criterion that allows one to reduce the problem of modularity of all deformations of $\rho_0$ to that of modularity of the *reducible* deformations of $\rho_0$. 

The condition $\dim_{F} \text{Ext}_{F}^{1}(G, \Sigma)(\chi_{0}, 1) = 1$ is (probably) essential (work in progress).

The unramifiedness of $\Psi$-condition can be replaced by demanding that $H^{2}_{c}(S, K_{f}, \mathbb{Z}_p)$ tors $= 0$. 

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We study the crystalline (or ordinary if $p$ splits) deformations of $\rho_0$. 

There exists a universal couple $(R_\Sigma, \rho_\Sigma : G_\Sigma \to \text{GL}_2(\mathbf{F}))$.

One gets a surjection $\phi : R_\Sigma \twoheadrightarrow T_\Sigma$.

Goal: Show that $\phi$ is an isomorphism.
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Method

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Ideal of reducibility

\[ l_{\text{re}} := \text{the smallest ideal } l \text{ of } R_{\Sigma} \text{ such that} \]

\[ \text{tr} \rho_{\Sigma} = \chi_1 + \chi_2 \pmod{l} \]

for \( \chi_1, \chi_2 \) characters.
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\( R_\Sigma/I_{re} \) controls the reducible deformations

**Key idea:** Reduce the problem to that of modularity of reducible lifts.
Commutative algebra criterion

Theorem (Berger-K.)

Let $R, S$ be commutative rings. Choose $r \in R$ such that \( \bigcap_n r^n R = 0 \). Let $A$ be a domain and suppose that $S$ is a finitely generated free module over $A$. Suppose we have a commutative diagrams of ring maps:

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\begin{array}{ccc}
R & \xrightarrow{\phi} & S \\
\downarrow & & \downarrow \\
R/rR & \xrightarrow{\overline{\phi}} & S/\phi(r)S.
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If $\text{rk}_A S/\phi(r)S = 0$, then $\phi$ is an isomorphism.

The rank condition can be replaced by a condition $rR \sim R/r$ and then the theorem gives an alternative to the criterion of Wiles and Lenstra.
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Applying the commutative algebra criterion

Corollary

Set $S = T_\Sigma$, $R = R_\Sigma$. 

Suppose $I_{re} = rR$ and 

If the map $\phi: R_\Sigma \twoheadrightarrow T_\Sigma$ induces an isomorphism $R_\Sigma/I_{re} \sim = T_\Sigma/\phi(I_{re})$, 

then $\phi$ is an isomorphism.

Upshot: 

To show $R_\Sigma = T_\Sigma$, it suffices to prove:

$I_{re}$ is principal, $R_\Sigma/I_{re} \sim = T_\Sigma/\phi(I_{re})$, i.e., that every reducible deformation of $\rho^0$ is modular.

Remark: The criterion removes the condition $p || B_2, \omega_k - 2$ from a modularity result for residually reducible Galois representations over $\mathbb{Q}$ due to Calegari.
Applying the commutative algebra criterion

**Corollary**

Set $S = T_\Sigma$, $R = R_\Sigma$. Suppose $I_{re} = rR$ and $\# T_\Sigma / \phi(I_{re}) T_\Sigma < \infty$. If the map $\phi: R_\Sigma \rightarrow T_\Sigma$ induces an isomorphism $R_\Sigma / I_{re} \cong T_\Sigma / \phi(I_{re})$, then $\phi$ is an isomorphism.

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Theorem (Bellaïche-Chenevier, Calegari)

If
\[ \dim F \Ext^1_{F[G\Sigma]}(\chi_0, 1) = \dim F \Ext^1_{F[G\Sigma]}(1, \chi_0) = 1, \]
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Theorem (Berger-K.)

Let \( A \) be a Noetherian local ring with \( 2 \in A^\times \). Set \( S = A[G_\Sigma] \). Let \( \rho : S \to M_2(A) \) be an \( A \)-algebra map with \( \rho = \rho_0 \mod \mathfrak{m}_A \).
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If \( A \) is reduced, infinite, but \( \#A/l_{re,A} < \infty \), then \( l_{re,A} \) is principal.
Goal is to show that $\phi : R_\Sigma/I_{re} \to T_\Sigma/\phi(I_{re})$ is an isomorphism.
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Two steps:

- Show \( \# R_\Sigma / I_{re} \leq \# \mathcal{O} / L \) – value (Iwasawa Main Conjecture - Rubin).
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- Show \( \# T_{\Sigma}/\phi(I_{\text{re}}) \geq \# \mathcal{O}/L \) – value (congruences - Berger).
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Then $I_{\text{re}}$ is principal for essentially self-dual deformations (Berger-K., 2011); Commutative algebra criterion still works;

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Higher-dimensional context

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One needs to prove $R_{\Sigma}/I_{\text{re}} = T_{\Sigma}/\phi(I_{\text{re}})$. This uses

- the Bloch-Kato conjecture for the module $\text{Hom}(\tilde{\tau}_2, \tilde{\tau}_1)$, where $\tilde{\tau}_j$ are the unique lifts of $\tau_j$ to $O$, 
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- the Bloch-Kato conjecture for the module $\text{Hom}(\tilde{\tau}_2, \tilde{\tau}_1)$, where $\tilde{\tau}_j$ are the unique lifts of $\tau_j$ to $\mathcal{O}$,
- Congruences among automorphic forms on higher-rank groups (results of Agarwal, Böcherer, Dummcigan, Schulze-Pillot and K. on congruences to the Yoshida lifts on $\text{Sp}_4$ allow us to prove that certain 4-dimensional Galois representations arise from Siegel modular forms.)
Another modularity result

\( N = \text{square-free}, \ k = \text{even}, \ p > k \geq 4, \ F = \mathbb{Q}. \) Assume that every prime \( l \mid N \) satisfies \( l \not\equiv 1 \mod p. \) Let \( f \in S_2(N), g \in S_k(N), \) \( \Sigma = \{l \mid N, p\}. \) Assume that \( \bar{\rho}_f \) and \( \bar{\rho}_g \) are absolutely irreducible.

**Theorem (Berger-K.)**

Suppose:
- \( \dim_F H^1_\Sigma(\mathbb{Q}, \text{Hom}(\bar{\rho}_g, \bar{\rho}_f(k/2 - 1))) = 1; \)
- \( R_{\bar{\rho}_f(k/2-1)} = R_{\bar{\rho}_g} = \mathcal{O}; \)
- the B-K conjecture holds for \( H^1_\Sigma(\mathbb{Q}, \text{Hom}(\rho_g, \rho_f(k/2 - 1)))). \)

Let \( \rho : G_{\mathbb{Q}, \Sigma} \to \text{GL}_4(\mathbb{Q}_p) \) be continuous, irreducible and such that:
- \( \bar{\rho}^{ss} \cong \bar{\rho}_f(k/2 - 1) \oplus \bar{\rho}_g; \)
- \( \rho \) is crystalline at \( p \) and essentially self-dual.

Then \( \rho \) is modular. More precisely, there exists a Siegel modular form of weight \( k/2 + 1, \) level \( \Gamma_0(N) \) and trivial character such that \( \rho \cong \rho_F. \)
Thank you.

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