L-functions, p-adic L-factors, and rational points:

**Known**: When analytic rank \( \leq 2 \)

**Theorem (Green-Zagier)**: When analytic rank = 2 = alg. rank \( \geq 2 \)

Use an auxiliary imag. quad. field \( K \rightarrow \mathbb{P}_K \subseteq E(K) \)

\[
L_K(E, s) = L(E, s) \cdot L(E, 2s, s)
\]

Choose \( K \) so \( s \) is odd

\[
\frac{L'(E, s)}{L(E, s)} \left( \begin{array}{c}
\frac{L(E, s) L(E, 2s, s)}{L(E, 2s) L(E, s, s)}
\end{array} \right)_{s=1}^{\infty}
\]

(see section below)

\[
\frac{L'(E, 1)}{L(E, 2, 1)} = \frac{\mathcal{O}}{O}
\]

\[
= \mathcal{P}_K \subseteq E(Q)
\]

\( \mathcal{H} = \) Hilbert class field of \( K \)

\[
X: \text{Gal}(K/\mathbb{Q}) \rightarrow \mathbb{C}^x
\]

\[
\Theta_{\mathcal{O}} = \sum_{\mathcal{O}(\mathfrak{P})} x(\mathfrak{p}) e^{2\pi i (\mathfrak{P}\cdot \mathfrak{M}) n} \quad \text{modular form of wt 2.}
\]

\[
L'(E \otimes \Theta_{x}, s)_{s=1} = \langle \mathcal{P}^x, \mathcal{P}^x \rangle
\]

\[
\mathcal{P}^x \in (E(H) \otimes \mathcal{O})^x \quad \text{if } x = 2
\]
Generalizations:

(a) Zhang: \( f \) modular form \( \omega \) wt \( k \). (even \( k \)). \( \text{Centr.} = k/2 \)
\[ K, \ x : \text{Gal}(H/k) \rightarrow C^\times \]
\[ L(f \circ \Theta^x, s) \]
\[ s = k/2 \]
"Heegner hypothesis" \( \Phi_{3k} = -1 \)

(b) Perrin-Riou: \( p \)-adic equivariant
\[ E/k \]
\[ \text{P-adic } L\text{-function} \]
\[ L(E, x, s) \]
interpolates \( x \) of finite order \( \ell \)

2 p-adic L-function
interpolates "Hecke trace" in a certain range \( \ell \)
Theorem (Perrin-Riou): \( \mathcal{O}_k^{(a)} \xrightarrow{\psi} \langle P, P_c \rangle \text{ pro-i} \).

(it remains an anti-cycl. var) This \( L \)-fct has 2 ramified, or for our \( Z_p \)-est.

**Question:** What if \( \chi \) is a Hecke character of \( K \)?

\[ \chi: K^*_{\mathfrak{m}} \rightarrow \mathcal{O}_k^* \] \( \chi \) has \( \psi \) and \( \chi |_{\mathcal{O}_k} = N^{-1} \chi \).

\[ \chi(x \mathfrak{m}) = \chi(x) x^{-\mathfrak{m}} \]

\( \chi \) has 

\[ \chi: \mathfrak{m} \rightarrow \mathcal{O}_k^* \]

\[ \mathfrak{m} \rightarrow \mathfrak{m}^* \]

\[ \text{Hecke char. of } K \rightarrow \mathfrak{m} \]

\[ \psi((\mathfrak{m})) = \psi \chi((\mathfrak{m})) \text{ Hecke char. of } K \rightarrow \mathfrak{m} \]

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Let \( z \) be a unique fixed point of \( \mathcal{K}^z \) on \( S \).

\[
S \rightarrow \frac{S}{\mathcal{K}^z} \sim \gamma_1(N)
\]

\( z \) \( \rightarrow \) \[ \mathcal{K} \in \gamma_0(N)(H) \]

\[
[z] \sim 100 \in \operatorname{Div}(X_0(N)) \stackrel{\text{Artin-Schreier map}}{\rightarrow} J_0(N)(H)
\]

\( (a) \) If \( h \) and \( z \) \( \leftrightarrow \) \( E \) \( \quad \) \( \text{send} \quad J_0(N)(H) \rightarrow E(H) \)

Call the point \( v \in E(H) \) \( \gamma_1 \) \( \rightarrow \) \( P_z \).

\[
P^x = \sum_{\sigma \in \operatorname{Gal}(H/k)} x^{\sigma} \cdot P_z \in (E(H) \otimes \mathbb{C})^x
\]

\( \gamma : \operatorname{Gal}(H/k) \rightarrow \mathbb{C}^x \)

\( \chi = 1 : E(H) \rightarrow E(K) \otimes \mathbb{C} \)

\[
(L(E), L(E,H)) \rightarrow \langle P \cdot P \rangle
\]

\( + \) \( \rightarrow P \in E(K) \quad \text{cc. act by } -1 \)

\( - \) \( \rightarrow P \in E(H) \)

This defines the "point \( \gamma_1 \), move to \( H \)."

\( (b) \) Zhang:

\( f \) \( \text{has wt} \ k = 2j \), \( x \) \( \text{finite order} \)

\( x \) \( \text{a \mathbb{Q} \text{-linear automorphism of } } X \)

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\( \text{by Deligne-Serre} \)

\( \mathcal{K} \rightarrow \mathbb{C} \)
\( f \) \& \( k \), \( x \) \& \( l \).

\[(3a) \quad l \leq k \quad \text{and} \quad \text{Sign} = -1.\]

Pick \( A = \text{CM}(l, k) \).

Elliptic curve:

\( f \mapsto (A \times W)^n \rightarrow A^t(A^t)^r \)

\( X_1(N) \)

\( \text{Hodge-Tate formula} \):

\[ L_X(F^2) = 1 \quad \text{for CM points twisted by } x. \]

Use Shimura-Maass operator to decay up the \( N \) \text{th} \& \( x \) \text{th}
ψ \text{ held at } 1 \text{ at } K

\chi = \psi_c

f = \Theta_{\psi c}

(f \circ \Theta, s) = L(\varphi_{\psi c}, s) \cdot L(\varphi, s\cdot r)