Automorphic forms and Galois representations:

$F$ number field

$l$ rational prime

$\overline{Q}_l \cong \mathbb{C}$

Def: ① A cuspidal autom. rep. $\pi$ of $\text{GL}_n(A_F)$ is algebraic
if $\text{H}_c(\pi_{\infty})$ lies in \( \left( \mathbb{Z}^n/S_n \right)^\text{Hom}(F, \mathbb{C}) \subset \left( \mathbb{C}^n/S_n \right)^\text{Hom}(F, \mathbb{C}) \).

② $\chi_{\overline{r}}: \text{Gal}(\overline{F}/F) = G_F \rightarrow \text{GL}_n(\overline{Q}_l)$ is a cont. ss.
rep. (l-adic rep.) we call $r$ algebraic if

a) $r$ is unramified a.e.

b) $\forall v \nmid l$, $r|_v$ is deRham

\[ \text{dim}_{\overline{Q}_l} (r \otimes \overline{r}_v (\text{cyclo})) \overset{G_F}{\longrightarrow} \text{mult. of } j \text{ in } HT(r)_v. \]

Example: $X/F$ smooth proj. variety.

$r = H^i(X \times F, \overline{Q}_l)$ is algebraic

\[ \text{mult. of } j \text{ in } HT(r)_F = \text{dim}_{\mathbb{C}} H^i_{\text{ét}}(X_{\overline{F}_v}, \mathbb{C}_{\overline{F}_v}). \]

Conjecture: (Langlands- Clozel- Fontaine- Mazur) There is

a bijection

\[ \begin{aligned}
&\{ \text{alg. cusp. auto} \} & \leftrightarrow & \{ \text{irred alg. } l\text{-adic rep.} \} \\
&\{ \text{reps of } \text{GL}_n(A_F) \} & \leftrightarrow & \{ G_F \rightarrow \text{GL}_n(\overline{Q}_l) \}
\end{aligned} \]
s.t. 1) $HC(\pi_{a0}) = HT(r_\lambda(\pi_{a0}))$.

2) A prime $\nu = \nu F_r$, $WD(r_{\nu}(\psi)) \rightarrow REC_{F_r} (\psi_{\nu})$.

3) $REC_{F_r}$ local Langlands, $\sigma \rightarrow Fr_{F_r}(\psi_{\nu})$.

4) $Fr_{F_r}(\psi_{\nu}) \leftrightarrow$ semi-simplify.

$Sp_2 : W_{F_r} \rightarrow SL_2(C)$ indecomposable

$\sigma \rightarrow \left( \begin{array}{cc} e^{\frac{1}{2} \psi_{\nu}} & * \\ 0 & e^{-\frac{1}{2} \psi_{\nu}} \end{array} \right)$

$(id, Sp_2) = j_{\nu} : W_{F_r} \rightarrow W_{F_r} \times SL_2(C)$

$v \times \lambda \begin{array}{c} r_\lambda(\psi) |_{W_{F_r}} \simeq WD(r_{\nu}(\psi)) \end{array}_{F_r} = j_{\nu}$.

n=1 this case is true. This is essentially class field theory.

Past n=2 one must impose some restrictions to get any

Theorems.

(1) Regularity

$\pi$ is called regular if $HC(\pi_{a0})$ consist of $n$
distinct integers $\lambda \in \I$.

$r$ is called regular if $HT(r_{\nu})$ consist of $n$ distinct
integers $\lambda \in \I$. 

If \( \pi \) is \textit{polarizable}

1. \( F \) is CM or totally real, \( F^+ \) maximal real subfield.
2. \( \exists \chi : \text{AF}^+/(F^+)^* \rightarrow \mathbb{C} \) s.t.

\[
\pi^c \cong \pi^c \otimes (\chi \circ N_{F/F^+}) \text{ det}
\]

and \( \chi_v(-1) \) is indep. of \( v \).

If \( \pi \) is \textit{polarizable} if

- \( F \) is totally real, \( r : G_F \rightarrow \text{GO}_n(\mathbb{Q}_2) \)
  
  multiplier totally even

  \( \mult(r)(c_v) = 2 \forall v \nmid \infty \)

  \( F \) is totally real, \( r : G_F \rightarrow \text{GSp}_n(\mathbb{Q}_2) \)
  
  multiplier totally odd

  \( \mult(r)(c_v) = 1 \forall v \nmid \infty \)

- \( F \) is imaginary, \( \exists \) a symm pairing \( \langle \cdot, \cdot \rangle \) s.t.

\[
\langle r(\sigma)x, r(\sigma \text{ conj} y) \rangle = X(\sigma) \langle x, y \rangle
\]

where \( X : G_{F^+} \rightarrow \mathbb{Q}_2^* \)

\( X(\mathfrak{c}_v) \) indep. of \( v \).

\textit{Theorem}: \( \chi \pi \) is a polarizable, res, alg, cusp. auto. rep.

of \( \text{GL}_n(\text{AF}) \), then \( \exists \) a polarizable, res, alg, \( \text{L} \)-adic rep. \( r_x(\pi) : G_F \rightarrow \text{GL}_n(\mathbb{Q}_2) \) s.t.

1. \( HT(r_x(\pi)) = HC(\pi^{\text{sm}}) \)

2. \( WD(r_x(\pi)|_{G_F}) F^{ss} = \text{rec}_{F_v}(\pi_v) \forall v \)

Moreover \( \pi_v \) is tempered \( V_v \).
This theorem is due to Shin, Chemerin - Harish, Caroaimi, etc.

Shin regular case:
\[ r_2(\mathbb{R}) \subset \text{coh. of } Y_{\text{Shimura variety}} \leftrightarrow G \]
\[ G(\mathbb{R}) \cong G(U(n-1,1) \times U(n)) \] 
\[ G(U(n,1) \times U(n+1,1)) \text{ if even.} \]

Other cases: \( r_2(\mathbb{Q}) \) is an \( l \)-adic limit and \( r_2(\mathbb{Q}) \otimes \mathbb{Q}_l \rightarrow G(U(n-1,1) \times U(n)) \)

**Theorem:** Suppose \( r \) is a polarizable regular alg. \( l \)-adic rep. \( r : G_F \rightarrow GL(n, \overline{\mathbb{Q}}_l) \). Assume further

a) \( I > \mathbb{Z}(n+1), \quad \mathbb{Z}(n+1) \notin F \).

b) \( \overline{r} = (r \text{ mod } I) \) irreducible in \( G_F/I \).

c) \( I \) unramified in \( F \), \( r|_{I^0} \) is crystalline.

\[ \forall \psi \in \mathcal{A} \text{ and } MT(r) \in \left( (\mathbb{Z}[0,1,2])^n/S_n \right)^{\text{Hom}(F, \overline{\mathbb{Q}}_l)} \]
then \( I \) a finite Galois \( E/F \) extension \( F/E \) and

a polarizable, crys, alg. cupro auto rep. \( r' \) of \( GL_n(M_{\mathbb{Q}_l}) \)

with \( r|_{I^0} \cong r'_{|_{I^0}} \).

Conditions a) - c) are true for most \( I \) in an ined. family.

**Theorem due to Baneret - Larche, Gee, Geraghty, T.**

Note one requires a finite base change \( F'/E \). However, for
Applications this is usually enough.

\[ \Rightarrow r \text{ pure} \]

\[ \Rightarrow r \text{ is part of a family as } l \text{-variety} \]

\[ L(r, s) = \prod \det (1 - r^{1/n} (Fract))^\# \text{ converges on some right half-plane, has meromorphic cont. to } \mathbf{C}^* \text{ and satisfies expected functional equation.} \]

\[ \Rightarrow \text{Sym}^{r-1}(H(E, \overline{\mathbf{Q}})) \]

\[ E/\mathbb{Q} \text{ elliptic curve} \]

\[ \Rightarrow 1 + p - \# E(F_p) \in [-2, 2] \]

\[ \frac{1}{\sqrt{p}} \text{ equidistributed wrt } \frac{1}{2\pi} \sqrt{4-2^2} dt \text{ (Sato-Tate)} \]

On the Galois side regular seems very restrictive, but on the automorphic side it seems to be very common.

Theorem: (Harris, Lan, T., Thorne) Suppose \( F \) is CM on \( \mathbb{R} \), real, and \( \pi \) is a regular alg. rep. of \( GL_n(\mathbb{R}) \).

Then \( \pi \) is a \( l \)-adic rep.

\[ \rho_{l}(\pi): G_F \to GL_n(\overline{\mathbf{Q}}_l) \]

s.t. for all but finitely many \( \nu \)

\[ \text{WD}(\rho_{l}(\pi)|_{F_\nu}) \equiv \text{rec}_{F_\nu} (\pi_\nu). \]

(This is most constructed in ch. 6 Shimura variety, but as a limit, etc it is believed they do most exist in coh. of Shimura variety!)}