Ad-up

$K/\mathbb{Q}$ imaginary quadratic field, odd class number

$\mathcal{O}_K =$ ring of integers of $K$

Bianchi group = $\text{PGL}_2(\mathcal{O}_K)$ (1892 L. Bianchi)

$\Gamma = \text{congruence subgroup of } \text{PGL}_2(\mathcal{O}_K)$

$\Gamma \leq \text{PGL}_2(\mathcal{O}_K) \leq \text{PGL}_2(\mathbb{C})$

$H = \mathbb{C} \times \mathbb{R}^3$ hyperboloid 3-space (symmetric space for $\text{PGL}_2(\mathbb{C})$)

$ds^2 = \frac{dx^2 + dy^2 + dr^2}{r^2}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z, r) = \left( \frac{(az + b)(cz + d) + ac r^2}{|cz + d|^2 + 1cr^2}, \frac{lad - bc) r}{|cz + d|^2 + 1cr^2} \right)$

Can view inside Hamiltonian quaternions:

$(z, r) \mapsto x + yi + rz + rj \in \left( \frac{-i}{\sqrt{2}} \right)$

$z = \begin{pmatrix} z \\ r \end{pmatrix}$

$\left( \begin{pmatrix} z \\ r \end{pmatrix} \right) \in \text{Mat}_2(\mathbb{C})$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z, r) = \left( \left( \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \right) \left( \begin{pmatrix} z \\ r \end{pmatrix} \right) \right) \left( \begin{pmatrix} \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \end{pmatrix} \left( \begin{pmatrix} z \\ r \end{pmatrix} \right) \right)^{-1}$

$\Gamma$ acts on $H$ via isometries; proper, discrete action.
\[ Y_\Gamma = \text{holomorphic 3-fold, non-compact} \]

\[ \text{finite volume.} \]

- \( Y_\Gamma \) has no complex structure - (main obstruction)

\( \Gamma \) totally real \( \Rightarrow \) nice - still have link to algebraic geometry here.

The simplest case beyond totally real is the case of \( \mathbb{R} \)-quadratic.

Modern History

- Marden
- his PhD student
- Grunewald
- Cremoña

More recent years:
- Calegari
- Venkatesh
- Bergamn
- Calegari - Gerovitch

Cohomology

\[ H^i(\Gamma, \mathbb{C}; \mathbb{C}) \cong \text{induced rep.} \]

\[ \text{induced rep. as Hecke module} \]
A cuspidal Bianchi modular form \( \psi \) of weight \( k \) and level \( \Gamma \) is a real analytic \( f : H \to \mathbb{C} \) satisfying

\[
\begin{align*}
&f(\frac{yz}{z^2}) = \text{Sym}^k \left( j(y, z) \right) f(yz) \quad \forall y \in \Gamma, z \in \mathbb{C} \\
j(y, z) = "cz+d" \in \text{Mat}_2(\mathbb{C})
\end{align*}
\]

- \( f \) satisfies certain diff. equation.
- \( f \) vanishes at cusps.

→ \( f \) has a Fourier-Bessel expansion
→ Hecke action via Fourier expansion
→ Also have Hecke action on cohomology: Using elements of
  \( \text{commensurator of } \Gamma (\text{PGL}_2(\mathbb{K})) \).

Talked about the dimension problem at Bristol.
- we do not know any formula for \( \dim S_k(\Gamma) \)
- recall \( H^1(\text{PSL}_2(\mathbb{Z}), M) \cong \frac{M}{M_{c_2} + M_{c_3}} \)

\[
\text{PSL}_2(\mathbb{Z}) = c_2 \times c_3.
\]

- Numerical data that suggest Bianchi modular forms are rare.
- Asymptotic results on growth of dimension

\[
(L, m) \quad \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \supset \cdots \quad \text{s.t. } \cap \Gamma_i = \{1\}.
\]
Connections with Motive:

\[
\lim_{i \to \infty} \frac{\dim S_2(\Gamma_i)}{[\Gamma_0 : \Gamma_i]} = 0.
\]

with 2 classical
\[\text{Jac}(X_\Gamma) \simeq A_f \times \cdots\]

One expects a similar correspondence here.

\[
\left\{ \text{Ell curves }/K \text{ with no CM by } K \right\}/\text{isogeny}
\]

of and \(K \text{ in } \mathbb{P}^1\).

\[? \rightarrow L\text{-factors agree?}\]

\[
\left\{ f \in S_2(\Gamma_0(N)) \text{ modular} \right\}
\]

integer Hecke eigenvalues

- Greenwald gave numerical data
- Cremona

Can't attack this with any of classic methods. This correspondence is believed based on computational evidence.

Remark: Sometimes \( f \in S_2(\Gamma_0(N)) \) satisfying condition

we want \( f \) \( A/K \) ab. surface with

\[\text{End}(A) = \text{quaternion algebra} & \quad \text{and } L(A/K, s) = L(f, s)^2\]

These adelic surfaces are often called "fake elliptic curves".

One must amend the above conjecture by most consistency module, forms that arise as base change, and not a \( \Theta \)-curve for, e., curve.
Asemi Conjecture:

\[
\{ \rho : G_k \to GL_2(\mathbb{Q}_p) \} \quad \xleftarrow{\text{Taylor, Harris,}} \quad \{ \text{eigenvalue system} \} \text{ in } H^2(\Gamma, E(C))
\]

\[
\{ \overline{\rho} : G_k \to GL_2(\overline{\mathbb{F}_p}) \} \quad \xleftarrow{\text{Serre's conj.}} \quad \{ \text{eigenvalue system} \} \text{ in } H^2(\Gamma, E(\overline{\mathbb{F}_p}))
\]

Because of torsion, there can be eigenvalue systems in

\[ H^2(\Gamma, E(\mathbb{F}_p)) \]

that are not mod p reductions of eigenvalue systems of \[ H^2(\Gamma, E(C)) \].

\[ H^1(\Gamma, E(C_k)) = \text{Tor} \oplus \text{Free} \]

There is a lot of torsion here!

Bergman-Venkataram give asymptotic results about torsion

Calegari-Venkataram give Jacquet-Langlands mod p.

\[
\begin{align*}
\Gamma & \quad \text{Bianchi} \\
\mathcal{S} & \quad \text{S-L} \\
\Gamma' \text{ cocompact arithmetic } & \quad PGL_2(C)
\end{align*}
\]

\[
\{ H^1(\Gamma, \mathbb{F}_p) \} \xleftarrow{\text{e.i.v. systems}} \{ H^1(\Gamma', \mathbb{F}_p) \} \text{ e.i.v. systems related...}
\]

Possible the torsion has nice p-finiteness, etc.