High families and Gross-Stark units over totally real fields:

\[ \mathbb{Q} \xrightarrow{\chi} \mathbb{C} \]

\[ \chi : (\mathbb{Z}/n\mathbb{Z})^* \rightarrow \mathbb{C}^* \quad \chi(-1) = -i \]

\[ H \xrightarrow{\chi} (\mathbb{Z}/n\mathbb{Z})^* \]

\[ L(\chi, s) = \sum_{n \geq 1} \frac{\chi(n)}{n^s} = \prod_p \left( 1 - \chi(p) p^{-s} \right)^{-1} \quad \text{for } \text{Re}(s) > 1 \]

Fix a prime \( p \) s.t. \( \chi(p) = 1 \) (i.e., \( p \) splits completely in \( H \))

\( \chi(n) = 0 \) if \( \gcd(a, N) \neq 1 \).

Let \( w : \mathbb{Q} \rightarrow \mathbb{Q}_p \) be the Teichmüller character.

\( p \)-adic \( L \)-function \( L_p(\chi w, s) : \mathbb{Z}_p \rightarrow \mathbb{Q}_p \) (values \( \chi \))

\[ L_p(\chi w, 1-k) = L(\chi w^{1-k}, 1-k) \quad \text{for } k \in \mathbb{Z}, \ k \geq 1. \]

\[ \text{LHS divisible by } p \]

\[ \text{LHS even if } \chi w^0(p) = 0, \ not \ 1. \]

\[ L_p(\chi w, 0) = L(\chi w^0, 0) \]

\[ L(\chi w^0, s) = (1 - \chi(p) p^{-s}) L(\chi, s) \]

\[ L(\chi w^0, 0) = (1 - 1) \cdot L(\chi, 0) = 0. \]
Def: \( \La_n(x) = \frac{L_p(x\omega, 0)}{L(x, 0)} \) \( \in \mathbb{E} \)

\( L(x, 0) \neq 0 \) since \( x(-1) = -1 \)

Let \( U = \{ u \in H^* : |1|w = 1 \ \forall \ w \in \mathbb{P} \} \). This is a finitely generated abelian group of rank \([H:G]/2\).

\( U_x = (U \otimes E)^{-1} = \{ u \in U \otimes E : \sigma(u) = u \otimes x^{-1}(\sigma) \ \forall \ \sigma \in G_{\mathbb{Q}} \} \).

\[ \dim \frac{U_x}{E} = 1 \]. Let \( u_0 \) be a generator of \( U_x \)

\[ \begin{array}{ccc}
U \subset H^* & \subset & G_{\mathbb{Q}}^* \\
\text{ker} & \rightarrow & \mathbb{Z} \\
\log_p & \downarrow & \\
\mathbb{Q} & \subset & \mathbb{Q} \\
\end{array} \]

Tensor with \( E \) to obtain:

\[ U \otimes E \xrightarrow{ord_p} E \xrightarrow{\log_p} \mathbb{Z} \]

Def: \( \La_y(x) = -\frac{\log_p(u_x)}{ord_p(u_x)} \leftrightarrow \neq 0 \)

Thm (Gross): \( \La_n(x) = \La_y(x) \).

Gross' proof is explicit, using formula for \( U_x \) in terms of Hasse sums.

Gross-Koblitz formula relates Hasse sums \( \Gamma_p \), p-adic \( \Gamma \)-function.
Ferrero-Heath-Burg theorem relates $P$ to $L_P$. This gives

the proof.

We'll give a different proof using Kelt's method.

One has the same conjecture for totally real fields. However,
since there is no explicit CFT here, Shpar's method does not
generalize, which is why this new method is nice.

Kelt's method, Step 1: Reformulation:

Kummer Theory: \[ \overline{U} = \prod_{\nu} \overline{\mathbb{Q}}^x : |\text{num}| = 1 \text{ for all } \nu \neq p^3 \]

\[ 1 \to \mu_{p^n} \to \overline{U} \to \overline{U} \to 1 \]

Let \( \mathcal{O} = \mathbb{Z}[x] \). Tensor the short exact sequence

by \( \mathcal{O}(x) : \)

\[ 1 \to \mu_{p^n} \otimes \mathcal{O}(x) \to \overline{U} \otimes \mathcal{O}(x) \to \overline{U} \otimes \mathcal{O}(x) \to 1. \]

Take \( G_\mathcal{O} - \text{cohomology:} \)

\[ 0 \to (U \otimes \mathcal{O})_{/p^n} \to H^1(G_\mathcal{O}, \mu_{p^n} \otimes \mathcal{O}(x)) \to H^1(G_\mathcal{O}, U \otimes \mathcal{O}(x))[p^n] \to 0. \]

Take \( \lim_{n \to \infty} \) and the $\otimes$-E.
\[ H_p(\mathbb{G}_O, E(x)(\Sigma)) \rightarrow "Image" \rightarrow 0 \]

\[ 0 \rightarrow (U \otimes E)^{\ast \ast} \rightarrow H'(\mathbb{G}_O, E(x)(\Sigma)) \rightarrow \bigoplus_{x \in \mathbb{G}_O} H'(\mathbb{G}_O, \widetilde{U} \otimes x(\Sigma)) \otimes \mathbb{E} \rightarrow 0 \]

\[ \bigoplus_{x \in \mathbb{G}_O} H'(\mathbb{G}_O, E(x)(\Sigma)) \rightarrow \bigoplus_{x \in \mathbb{G}_O} T_p(\mathbb{H}(\mathbb{G}_O, \widetilde{U} \otimes x(\Sigma)) \otimes \mathbb{E} \rightarrow 0 \]

\[ H_p(\mathbb{G}_O, E(x)(\Sigma)) = \sum_{x \in \mathbb{G}_O} H'(\mathbb{G}_O, E(x)(\Sigma)) : \text{res}_{\mathbb{G}_O} x = 0 \quad \forall \neq p \]

"Image" \in \ker \alpha = T_p (\ker (H'(\mathbb{G}_O) \rightarrow \bigoplus_{x \in \mathbb{G}_O} H'(\mathbb{G}_O)) \otimes \mathbb{E} = 0

\[ \Rightarrow (U \otimes E)^{\ast \ast} \cong H_p(\mathbb{G}_O, E(x)(\Sigma)) \]

\[ U_x \rightarrow [Y_x] \]

**Local Restriction:**

\[ H_p'(\mathbb{G}_O, E(x)(\Sigma)) \rightarrow H'(\mathbb{G}_p, E(x)(\Sigma)) = H'(\mathbb{G}_p, E(\Sigma)) \]

\[ H_p'(\mathbb{G}_O, E(x)) \rightarrow H'(\mathbb{G}_p, E(x)) = H_{\mathbb{G}_p}(\mathbb{G}_O, E) \]

Consider

\[ H_p'(\mathbb{G}_O, E(x)^{-1}) \rightarrow H'(\mathbb{G}_p, E(\Sigma)) = H'(\mathbb{G}_p, E) = H_{\mathbb{G}_p}(\mathbb{G}_O, E) \]

\[ \sum_{x \in \mathbb{G}_O} H'(\mathbb{G}_O, E(x)^{-1}) : \text{res}_{\mathbb{G}_O} x = 0 \quad \forall \neq p \]