Goal of this series of talks: Explain a construction of
\( p \)-adic families of holomorphic and nearly holomorphic
Siegel modular forms. (The techniques work in great
generality, just restricting to Siegel modular forms to
simplify the notation.)

Example of a \( p \)-adic family:
\[
\begin{align*}
5p(s) &= \frac{g((1+p)^s-1)}{(1+p)^s - (1+p)} \\
\text{where } g(T) &\in \mathbb{Z}_p[[T]] \mod \Lambda. \\
h((1+p)^s-1) \\
h(T) &= 1+T - (1+p).
\end{align*}
\]

\[
E = \frac{1}{2} \frac{g((1+p)(1+T)^{-1}-1)}{h((1+p)(1+T)^{-1}-1)} + \sum_{d=1}^{\infty} \left( \sum_{\phi|d} \phi^{-1} \frac{(1+T)^{\rho d}}{p^{\rho d}} \right) q^n \in \Lambda[[q]]
\]

\( p \) only for simplicity.

\( d = \omega(d)(1+p)^{\rho d} \)

\( \mathbb{Z}_p^\times = \mathbb{Z}_{(p)} \times (1+p)^{\mathbb{Z}} \)

Put \( 1+T = (1+p)^k \) \( k \geq 2 \).

Resulting \( q \)-expansion is that of \( E^\text{ord}_k \cdot (q^{-k}) = (\text{i.e., has}
Up: \text{eigenvalue } = 2.)

\( E \) is a \( p \)-adic family in the sense that its \( q \)-expansion
coffs are \( p \)-adic analytic functions and its specializations
at "classical weights" are \( q \)-expansions of classical
modular forms.
Weight space: \( W = \text{Hom}_{\mathbb{C}^*}(\mathbb{Z}_p^\times, \mathbb{C}^*) \to \mathbb{Z} : x \mapsto x^k \)

\[
\text{Hom}_{\mathbb{C}^*}(\mathbb{Z}_p^\times, \mathbb{C}^*) \xrightarrow{\phi} \mathbb{Z} \quad \text{such that} \quad \sum_{a \in \mathbb{C}^*} \mathbb{Z}_p \times \{ x \in \mathbb{C}^* \mid 1 < |1 + ap^{-1}x| < 2 \} \quad \phi(\mathbb{Z}_p^\times) = \mathbb{Z}^n \mathbb{C}^*.
\]

\( k \in \mathbb{Z}_{>0} \), \( U \ni x \in \text{aff} \in W \) (affine nbh of \( x \))

Ex: \( U = \mathbb{C}^* \times \{ x \in \mathbb{C}^* \mid 1 < |1 + ap^{-1}x| < 2 \} \).

\( \mathcal{O}(U) \): analytic functions on \( U \cap \mathbb{C}^* \sim \mathbb{C}^* \).

Then consider formal series

\[
F = \sum_{n=0}^{\infty} a_n q^n, \quad a_n \in A \quad A/\mathcal{O}(U) \text{ finite regular ring}
\]

\( \phi \in \text{Hom}_{\mathbb{C}^*}(A, \mathbb{C}^*) \) s.t. \( \phi|_{\mathbb{C}^*} = \text{specialization at } \alpha = (1+p)^k \)

require the resulting \( q \)-expansion

\[
F_\phi = \sum_{n=0}^{\infty} \phi(a_n) q^n
\]

to be the \( q \)-expansion of a \( k \)-modular form.

Question: Does any form \( f \) of \( wt \ k \) fit into such a family?

(eigenform \( f \) \( \to \) eigenfamily)

\[
f, \quad E^k((1+p)^{-1}(1+\tau)) = G
\]

\( E^k = \text{h}((1+p)(1+\tau)^{-1}) E \)

\( wt \ k = 0 \) specialization of \( E^k = \text{const.} \)
Then the weight $k$ specialization of $G$ is $(\text{const}) f$. This shows the answer to the first question is yes, but this does not answer the second question. The second question is harder! This one is most easily dealt with by focusing on families of fixed finite slope ($|\text{Up-}\text{eigenvalues}| = \text{constant} \neq 0$).

Ex: $E$ \text{ Up-}\text{eigenvalue } \equiv 1. \\

For finite slope families:

$$M^0(A) = A\text{-module of } A\text{-families of slope } r = \text{ord}(\text{Up-}\text{eigenvalue}) < \infty$$

This is a finite (torsion-free) $A$-module. One can act on this by Hecke-adj. $1A$. Can find eigenprojectives.

Note: When $r = 0$, this is the ordinary case and has been developed by Hida. $r > 0 \Rightarrow$ Coleman…

By work of Ash--Atkin--Igusa, when can construct a projective to slope-$r$ forms over some nod $U \in \mathbb{A}$. If we assume if has slope-$r$, we can put it into a family of slope-$r$ forms and then into an eigenfamily.

Can imagine generalizing this:

(i) def. of families of forms \checkmark

(ii) construct finite slope projective \checkmark
(iii) put a given finite slope form into an analytic family.

Ideally, family should have as much freedom as weight allows.

Ex: $F$ totally real deg $d$, $H.M.F. = (x_1, ..., x_d) \in \mathbb{Z}^d$

problem $E_S = (x, ..., x)$

As can only move $E_S$ in one direction.

Same problem for Siegel modular forms.

$$E^x = \sum_{n=1}^{\infty} a(n) q^n \quad a(n) \in \Lambda$$

where analytic form.

$$\frac{\sum n a(n) q^n + \frac{1}{2\pi i} E^x ((1+p)^{-2}(1+q))}{\sum n a(n) q^n}$$

Specialize at $w k = \text{get } a \cdot q\text{-exp. } (k > 2)$

$$\frac{1}{2\pi^2} y^{2-k} \frac{\partial}{\partial z} (y^{k-2} E^x_{k-2})$$

$\text{Maszik Shimura differential operator; take } w \text{ to } w + k + 2$

$k = 2$: (multiple) $E_2^{\text{cusp}}(z) = E_2(z) - E_2(pz)$.

This is an example of a family of nearly holomorphic modular forms of slope $2$.

We deal with part (iii) by restricting nearly holomorphic families from a larger group.

Ex: $\xi \cdot \xi \rightarrow \xi^2$
Angel Modular Forms:

The symplectic (similitude) group:

\[ n > 1 \]
\[ J = J_n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]
\[ G_{Sp_{2n}} = \{ g \in GL_{2n} : gJg^T = \lambda_g J \} \quad \lambda_g \in \mathbb{G}_m \]

\[ \lambda : G_{Sp_{2n}} \rightarrow \mathbb{G}_m \quad \text{similitude character} \]

\[ g \mapsto \lambda_g \]

\[ Sp_{2n} = \ker \lambda. \]

**Ex:** \[ n = 1 \]
\[ G_{Sp_2} = GL_2 \]
\[ Sp_2 = SL_2. \]
\[ \lambda = \det. \]

\[ Sp_{2n}(\mathbb{R}) \]

\[ K_0 = K_{0, n} = \{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} : Sp_{2n}(\mathbb{R}) \} \quad \text{maximal compact} \]

\[ K_0 \xrightarrow{\sim} U(n) \]
\[ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mapsto u(B) = A + iB. \]
\[ u \in K(A, B). \]

**Ex:** \[ n = 1 \]
\[ K_0 = SO_3(\mathbb{R}) \approx U(1) \]
\[ \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \mapsto e^{i\theta}. \]

The half-space \[ \mathcal{H} = \mathbb{H} = \{ Z \in M_n(\mathbb{C}) : \operatorname{tr} Z = 0, \quad X > Y, \quad Y > 0 \}. \]
\[ \text{Span}(\mathbb{R}) \text{ acts on } S^n \text{ via } \gamma \]

\[ (A \ B) \cdot \gamma = \gamma(Az + B) = (Az + B)(Cz + D)^{-1}. \]

- transitive action
- \( K_0 = \text{stabilizer of } i \in S \)
- \( \text{Span}(\mathbb{R})_{K_0} \to S, \quad \gamma \mapsto \gamma(i) \).

\[ \text{The automorphism factor:} \]

\[ j : \text{Span}(\mathbb{R}) \times S \to U(n)_\mathbb{C} = GL_n(\mathbb{C}) \]

\[ j(\gamma, z) = Cz + D, \quad \gamma = (A \ B) \text{.} \]

Weights: complex algebraic representations of \( GL_n(\mathbb{C}) \). (p.v)

Indexed by highest weight vector upper triangle Borel

\[ \rho \leftarrow (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n \]

\[ \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \]

action on \((c_1, \ldots, t_n)\) on highest weight vector \( \Pi(c_1) \).

Siegel modular forms: (p.v) weight

\[ \Gamma \subseteq \text{Span}(\mathbb{Z}) \text{ a congruence subgroup (i.e., } \Gamma \text{ contains } \sum_{\gamma \in \Gamma} \text{ mod } N \text{ for some } N > 1). \]

\[ C^\infty - \text{case: smooth functions } \mathcal{F} : S \to V \text{ s.t.} \]

\[ (\mathcal{F}, \gamma) := \rho(\gamma j) \mathcal{F}(j(\gamma z)) \quad \forall \gamma \in \Gamma. \]
Ex: \( n=2 \) 
\[ \rho = \text{det}^k = x^k \]

Recover usual def.

Holomorphic case just look at holomorphic functions instead of smooth.

(we ignore any growth condition here)

Fourier expansion:
\[ \Gamma \supseteq \Gamma(N) \Rightarrow \Gamma \supseteq \left( \frac{1}{2} \mathbb{Z} \right)^N \]
\[ \mathfrak{L} = \mathfrak{L}_\mathbb{N} \supseteq \frac{1}{2N} \mathbb{M}_n(\mathbb{Z}) \]

Hecke operators: Can find classical info. in Andrianov's books.

\[ D = \begin{pmatrix} d_1 & \cdots & \cdots & d_n \\ \vdots & \ddots & \ddots & \vdots \\ \cdots & \cdots & \ddots & \cdots \\ d_n & \cdots & \cdots & d_1 \end{pmatrix} \quad \text{di integers } d_1,d_2,\ldots,d_n \]

\[ T_D = \Gamma(D^{-1}) \Gamma = \prod \Gamma \alpha_i \]

\[ T_D f = \left( \prod \alpha_i^{k_i - (n+1)} \right) \sum_i \phi_i \alpha_i. \]

Ex: \( n=2 \).

\[ T_\mathfrak{L} = \Gamma \left( \mathfrak{L}^\mathbb{Z} \right) \Gamma \quad \mathfrak{L} \in \mathbb{N}. \]
\[ T_\ell f = \ell^{2\ell - 2} f(\ell^2 z) + \frac{1}{\ell^2} \sum_{\alpha = 1}^{\ell^2} f \left( \frac{z - \alpha}{\ell^2} \right). \]