Algebraicity of $L$-functions for $GL_2$:

Goal: $f \in S_k \left( \Gamma_1(N), \chi \right)_{new}$ (k even)

There exist period $\Omega_f \in \mathbb{C}$ such that the following holds:

If $\eta$ is a Dirichlet character of cond. $M$ with $\eta(-1) = (-1)^k$ then

$$\Omega_f \sum_{n=1}^{\infty} \frac{a_n \eta(n)}{n^s} \chi(n)$$

with $0 \leq j \leq k-2$, and $\Omega_f(n) = \sum_{d|n} \frac{\chi(d)}{d}$.

$$f = \sum a_n q^n$$

$$L(f, \eta, s) = \sum_{n=1}^{\infty} a_n \eta(n) n^{-s} \text{ for } \text{Re}(s) \text{ large}.$$ Note this does not converge in the range $1 \leq s \leq k-1$. As $L(f, \eta, s)$ is defined by analytic continuation.

Key formula:

$$\Gamma(s) L(f, s) = \int_0^\infty \int_0^\infty f(y) y^s \frac{dy}{y} \quad \text{for } \text{Re}(s) \text{ large}.$$
\[ \sum \int_0^\infty a_n e^{-\pi n y (iy)^{s-1}} \, dy \]

\[ = \sum a_n \int_0^\infty e^{-\pi n y y^{s-1}} \, dy \]

\[ = \ldots \text{ gives a nice formula for Re}(s) \text{ large.} \]

One still must deal with \( \int_0^1 \) when \( \text{Re}(s) \) is not necessarily large.

**Key point:** \( \int_0^1 f(z) \, dz \leftrightarrow \int_0^\infty g(z) \, dz \quad g(z) = f \left( \frac{1}{z} \right) \).

\[ \text{converge for all } s. \]

**Main point:** Get an expression for \( \frac{\Gamma(s)L(f,s)}{(2\pi)^s} \) that is valid for all \( s \), via this trick. This gives a formula for \( L(f,s) \) with \( s \) in the range of interest.

**Interpretation:** Replace the upper half-plane integral \( \int_0^\infty \), look at the integral on \( \Gamma_i(\mathbb{N}) \), \( y = f(z) \, dz \).

(Assume \( k = 1 \) for this part...)

\[ \frac{L(f,s)}{2\pi i} = \int Y f(z) \, dz \]

where \( Y \) is the image in \( \Gamma_i(\mathbb{N}) \) of path from 0 to \( 1 \).

This integral actually converges. The integral \( \int_Y \) evaluates \( L(f,s) \) at the point of interest.

For twisted \( L \)-function:
Replace the path from 0 to i∞ by a sum of such paths:

\[ \sum_{a=1}^{n} \left\{ \frac{a}{n} \to i\infty \right\} \eta(a) \to \text{project to } \Gamma_1(n) \text{ to path from } \frac{a}{n} \to i\infty \int \text{ integrate w.r.t.} \]

**Summary:** For each \( \eta \) we get a path \( X(\eta) \) in the upper half plane as above, project to \( \Gamma_1(n) \) and get a compact path \( Y(\eta) \) on the modular curve. Integrate w.r.t. to get the \( L \)-value. To do higher weight the differential form must be modified.

**Remark:** In general, if \( a, b \) are elements in \( \mathbb{P}^1(\mathbb{Q}) \) one can make the same recipe with a path from \( a \to b \) in \( \mathbb{A} \). This is usually called the modular symbol \( [a, b] \). One can then integrate w.r.t. on the image of this path in \( X_1(N) \).

**Observe:** Given any differential \( \omega \in H^0(X_1(N), \Omega^1) \), we can integrate \( \int_{Y(\eta)} \omega \)

get a functional on \( H^0(X_1(N), \Omega^1) \). This defines \( Y(\eta) \) as an element of \( H_1(X_1(N), \mathbb{Q}) \).

Elementary Hodge theory: Each \( [a, b] \to \eta(a, b) \) defines an element of \( H_1(X_1(N), \mathbb{R}) \) (\( \mathbb{R} \)-dual of \( H^0(X, \Omega^1) \)).
Theorem (Manin-Drinfeld): \( \gamma(a,b) \) lies in \( H_1(X_1(N), \mathbb{Q}) \).

Assume this for now and show how this implies the main theorem \((K=\mathbb{Q})\).

Example: \( N = 11 \)

\( X_0(11) \) has genus \( 2 \) with two cusps \( 0, \infty \).

\[
H_1(X_0(11)) = \mathbb{Z} \oplus \mathbb{Z}
\]

\[
H_1(X_0(11), \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}.
\]

Pick this decomposition according to the \( \pm \) eigenspaces of complex conjugation. Pick generators \( \gamma^\pm \) of each eigenspace.

\[
\begin{align*}
\gamma_+ & = \gamma_+^1 \oplus \gamma_+^2 \\
\gamma_0, \gamma_\infty & = \gamma_0^- \oplus \gamma_\infty^-
\end{align*}
\]

Utilize the unique \( \Phi \) from \( \Phi^+ \) on \( \gamma^\pm \mapsto \Phi^\pm \in \mathbb{C}^\times \).

For any other element \( \gamma \in H_1(X_0(11), \mathbb{Q}) \) have

\[
\gamma = \gamma^+ \oplus \gamma^- = c^+ \gamma^+ \oplus c^- \gamma^-
\]

with \( c^\pm \in \mathbb{Q} \).

Now \( \int \omega \gamma = c^\pm \int \omega \gamma^\pm = c^\pm \int \frac{\gamma^\pm}{\gamma^\pm} \).

Apply this to image of \( \gamma(\eta) \in H_1(X_1(N), \mathbb{Q}) \) we get that

\[
\int \omega_\gamma = c^+(\eta) \delta^+ + c^-(-\eta) \delta^-
\]

\( \gamma(\eta) \)

with \( c^\pm(\eta) \in \mathbb{Q} \).