Minimal modularity lifting theorems over imaginary quadratic fields:

(work in progress w/ Frank Calegari)

Introduction:

Let \( f \in \mathbb{S}_k \left( \Gamma_0(N), \mathbb{Y} \right) \), \( k \geq 1 \) newform, \( E_f = \mathbb{Q}(a_n : n \geq 1) \)
where \( f = \sum_{n} a(n) q^n \). For each \( p \) a prime of \( E_f \) there exists a
continuous irreducible Galois rep.

\[
\rho_p = \rho_p(f) : G_{\mathbb{Q}} \rightarrow GL_2(E_{f,p})
\]
unramified at all \( l \not| Np \) when \( p \nmid Np \), and

\[
\det \left( x - \rho_p(Frob_p) \right) = x^2 - a_p x + \gamma_p x^{p-1}.
\]

Question: What cont. reps. \( r : G_{\mathbb{Q}} \rightarrow GL_2(\mathbb{Q}_p) \) arise in
this fashion?

Theorem (Khare-Wintenberger, Kisin): ("Fontaine-Messing conjecture")

Under some mild hypotheses, \( r \) is modular \iff

(1) \( r \) is unramified at all but finitely many \( l \)

(2) \( \rho_{G_{\mathbb{Q}_p}} \) is "deRham"

(3) \( \left( \det r \right) \text{ (complex. conj.)} = -1 \) "odd"

deRham?: (1) Let \( \varepsilon : G_{\mathbb{Q}_p} \rightarrow \mathbb{Z}_p^\times \) be the \( p \)-adic cyclotomic char.

This is deRham, in fact, \( \varepsilon^k \) is deRham \( \forall k \in \mathbb{Z} \).

This \( k \) is called the Hodge-Tate weight.

(2) Let \( \Pi \) be a Hida family, \( \mathcal{P} \in \text{Spec}(\overline{\mathbb{Q}}) \) an arithmetic
point \( \leftrightarrow S_k \left( \Gamma_0(Np^r), \mathbb{Y} \right) \), \( k \in \mathbb{Z} \)

Then \[
\rho_{\mathcal{P}} |_{U \mathbb{Q}_p} \sim \left( \begin{array}{cc}
\varepsilon^{k-1} & * \\
0 & \varepsilon^k
\end{array} \right)
\]

is deRham and the Hodge-Tate wt(\( \varepsilon \)) = \( \sum \), \( k-1 \).
To any $n$-dimensional de Rham rep. of $G_F$ one can associate $n$ integers $(h_1, \ldots, h_n)$ "Hodge-Tate" numbers.

Say $r$ is regular if the $h_i$ are pairwise distinct. For example, $f$ with $r_F$ is regular $\iff k \geq 2$.

**Conjecture (Langlands, Fontaine-Mazur):** $F$ number field, $n \geq 1$.

There exists a bijection $s.t. \ L(\tau, s) = L(\tau, s)$

$$\begin{cases}
\text{auto. rep. of } & \text{compatible systems of } \\
\text{Gl}_n/F & \text{mod. alg. repo.} \\
\text{which are cusp.} & \text{mod. alg. repo.}
\end{cases} \iff \begin{cases}
R_p : G_F \to \text{Gl}_n(C_F) \\
\text{cusp. alg.}
\end{cases}$$

$r$ is algebraic iff infinitesimal char. of $\pi_{deR}$ is alg., i.e., equals that of a finiteness mod. rep.

$r$ is algebraic iff

1. $r|_{G_F}$ is unramified for all but finitely many $\nu$.
2. $r|_{G_F}$ is de Rham $\forall \nu|p$.

**$F = \mathbb{Q}$, $n = 2$:** $\pi$ cusp. auto. rep.

- $\pi \leftrightarrow f$ of wt $k \geq 2$ $\exists R_p(\pi)_p$ exist
- $\pi \leftrightarrow f$ of wt $k = 1$ $\exists R_p(\pi)_p$ exist
- $\pi \leftrightarrow$ Maass eigenforms $\Delta g = \frac{1}{4} g$ Conjecturally these correspond to even reps.

Want to discuss method for associating an auto. form to a Galois rep. Most successful method is Taylor-Wiles method.
b) F is either CM or totally real, n general

If F is CM, we must assume \( \pi \otimes c = \pi^v \otimes \chi \) \( c = \text{c.c.} \).

If F is totally real, we must assume \( \pi = \pi^v \otimes \chi \).

If \( \pi \) is regular, then \( \{ R_{p}(\pi) \} \) exists (Harris-Taylor, Shin, Chenursin-Harris).

For certain non-regular \( \pi \), \( \{ R_{p}(\pi) \} \) was constructed by Wushi Goldberg. These have HT-heights \( h, h_2, \ldots, h_{i, h = h_{i+1}, h_{i+2}, \ldots, h_n} \).

For regular \( \pi \): \( R_{p}(\pi) \) sits naturally in the étale cohomology of a Shimura variety

\[ \pi^{\text{cusp}} \otimes R_{p}(\pi) \rightarrow H^1(S_{\chi}, \mathbf{Z}) \]

For non-regular \( \pi \): \( R_{p}(\pi) \) are constructed by finding congruences between \( \pi \) and other regular \( \pi' \)'s in the coherent cohomology of Shimura varieties.

c) F quadratic imag., \( n = 2 \), \( \pi \) cuspid. reg. algebraic, \( \omega_{\pi} \circ c = \omega_{\pi} \) \( (\omega_{\pi} = \text{central char.}) \). Then \( R_{p}(\pi) \) exists (Taylor, Harris-Ascher-Taylor).

Automorphic representation in cohomology:

\[ F = \mathbb{Q}, n = 2, f \text{ of wt } 2, \pi \text{ corresponding auto. rep.} \]
\[ \pi \text{ contributes to } H^1(X, \mathbb{C}) \text{ and to } H^0(X, \mathbb{C}) \]
\[ f \mapsto \pi. \pi \text{ contributes to } H^0(X, \omega) \text{ and } H^1(X, \omega) \]

Serre-duality.

\[ F = \text{quad. imag., } n = 2, \pi \text{ cusp. reg. alg., trivial inf. char., } \]

\[ U \subseteq G_{L_2}(\mathbb{A}_F^\infty) \rightarrow X_u = G_{L_2}(F)/G(M_2)/U \cdot U(1,1) \mathbb{C}^* \]

3-dim manifold.

\[ \pi \text{ contributes to } H^0 \text{ cusp } (X_u, \mathbb{C}) \text{ and } H^0 \text{ cusp } (X_u, \mathbb{C}) \]

related by Poincaré duality.

Malcev Theory:

F number field, \( F \rightarrow GL_2(\mathbb{F}_p) \) abs.

Fix \( \chi : GL_2(\mathbb{F}_p) \rightarrow \mathbb{Z}_p^* \) lifts det F.

\[ S \subseteq \{ \text{v}_1, \ldots, \text{v}_l \} \subseteq \text{V} : F|_{\text{GF}_v} \text{ is ramified?} \]

finite set of places of F.

\[ \text{For each } \tau : F \rightarrow \overline{\mathbb{Q}}_p \text{ fix } k_{\tau_1}, k_{\tau_2} \in \mathbb{Z} \text{ (Hodge-Tate wts).} \]

(Assume p splits all \( \text{v}_1, \ldots, \text{v}_l \) in F. For each \( \text{v}_i \), fix \( k_{\text{v}_i} \).

\( k_{\tau_1}, k_{\tau_2} \in \mathbb{Z} \) can do this instead of \( \tau \)'s...)

Assume \( p \gg 0 \).

Def \( \mathbb{F}^\tau (k, s) : \text{ART}^{\mathbb{Z}_p} \rightarrow \text{Sets} \)

\[ A \rightarrow \text{fr} : \mathbb{F}_p \rightarrow GL_2(A) \]

\[ r \text{ mod } ma \leq r \]

\[ \text{det } r = (\mathbb{F}_p^\tau \rightarrow \mathbb{Z}_p^* \rightarrow \mathbb{A}^1) \]

\( r \) ramified outside \( s \)

\( F|_{\mathbb{G}_m} \) cusp. with H.T.

wts \( k_{\tau_1}, k_{\tau_2} \).

This is pre-represented by a complete Noetherian local \( \mathbb{Z}_p \)-alg. \( \mathbb{R}^{\text{univ}}(k, s) \).

Since we have fixed weight and level, we expect \( \mathbb{R}^{\text{univ}}(k, s) \).
to have only finitely many $\mathbb{Q}_p$-points. Write
\[ \mathbb{Z}_p[x_1, \ldots, x_g] \xrightarrow{\sim} R_F^\text{univ} (\kappa, S) \]
\[(f_1, \ldots, f_r)\]
with $g, r$ minimal. Then Krull-$\dim R_F^\text{univ} (\kappa, S) \geq 1 + g - r \geq 1 - S$

\[
\dim \text{Gal}(F/F_0) \]

where $S = \sum_{v \mid oo} (\text{ad}^0 F)^v G_F - \sum_{i=1}^g (1 - S_{K_i})$.

Call $S = S(\kappa, F_{1,0})$ the "Taylor-Wiles defect".

- $S > 0$. The Taylor-Wiles method works only for $S = 0$.
- $S$ seems to predict the number of cohomological degree in which the $\kappa_i$'s that should correspond to reps $\rho$ of type $(\kappa, S)$ appear. ($= S + 1$)

**Example:**

1. $\mathbb{R}$ is a tot. real field, $K_i_1 > K_i_2 \forall i$, $F$ is tot. odd.

\[
\begin{aligned}
\text{dim} (\text{ad}^0 F)^v G_F = \begin{cases} 
3 & \text{if } v \text{ complex} \\
3 & \text{if } v \text{ real, } \det_G(v) = 1 \\
1 & \text{if } v \text{ real, } \det_G(v) = -1 
\end{cases}
\end{aligned}
\]

\[ S = 1 + \ldots + 1 - (1 + \ldots + 1) = 0 \]

\[ \overline{[F:A]} \]

$S = 0 \iff$ these conditions hold.

2. $F = \mathbb{Q}$ $F$ is odd, $K_1 = K_2 = 0$ ($\iff$ at 1 forms)

\[ S = 1 - 0 = 1 \]
③ $F$ quad imag., $K_{i,1} > K_{i,2} = \langle 1, a \rangle$ ($\rightarrow$ $\mathfrak{c}$ reg alg., \\
for $i=1,2$ \\
twist, infr. char. \\
$S = 3 - (1+1) = 1.$

Crystalline reps. $r : G_F \to GSp_4(\mathbb{Q}_p)$ odd similitude char.
H.T. weights $(h_1 > h_2 > h_3 > h_4)$ $h_1 + h_4 = h_2 + h_3$.

- **regular case** $S = 0$ $\mathfrak{c}$’s appear in one degree (Betti, coh.)
③ $(h,h,0,0)$ $h > 0$ $S = 1$ $\mathfrak{c}$’s appear in $H^0(X, W, H)$ \\
and $H^1(X, W, H)$.
④ $(h,0,0,-h)$ $h > 0$ $S = 1$ $\mathfrak{c}$’s appear in $H^1(X, W, H)$ \\
and $H^2(X, W, H)$.

**Main Result:**
Suppose we are in one of the 4 cases ①-④ where $S=1$. To 
be concrete, ②.
$F$ quad imag. $r : G_F \to GL_2(\mathbb{Q}_p)$
$p$ split as $v_1, v_2$
- unram, a.e.
- crystalline H.T at $(1,0)$ at $v_1, v_2$
- $F$ is $\mathfrak{c}$ irreducible
- $r$ is minimal deformation of $F$.

Suppose following condition hold:
I. (Serre’s criterion): $U < GL_2(\mathbb{A}_F^\infty)$ compact open, let $\mathbb{T}(U) \subset H^2(X_0, \mathbb{Z}_p)$
$Z_p$-alg. gen. by $T_v$ at good primes $v$. Let 
$M = (p, T_v - F(\text{Frob}_v) \text{ v good place}) \subset \mathbb{T}(U)$. Then 
$M$ is a proper maximal ideal of $\mathbb{T}(U_{\text{min}})$.
II. (Existence of Galois reps.): $\forall U \subset U_{\text{min}} \exists r^\text{mod} : G_F \to GL_2(\mathbb{T}(U)_{\text{mod}})$ 

satisfying nice properties.
(Cohomology vanishing integrally): \( H^i(X_u, \mathbb{Z}_p)_m = \{0\} \) \( i \neq 1, 2 \).

(Duality): \( \dim H^i(X_u, \mathbb{Q}_p)_m = \dim H^2(X_u, \mathbb{Q}_p)_m \)

Then \( R^\beta \cong T(U^{\min})_m \) and act freely on \( H^2(U^{\min}, \mathbb{Z}_p)_m \).