Pencils of Selmer Groups Part II:

K imag. quadratic

V p-adic rep. of \( G_K \), irreducible, cont., fin. dim., motivic, pure,

\[ V^c = V^c(1) \quad (c = \text{nontrivial auto. of } K, \quad V^c = V^{\circ c}). \]

Would like to show \( L(V,0) = 0 \Rightarrow H^1_f(K, V^c(1)) \neq 0 \).

We consider deformations of \( \varepsilon \otimes V \otimes 1 \). (Would like to look at \( \varepsilon \otimes V \), but need to throw on \( \otimes 1 \) so the rep. satisfies the self-dual condition given above.)

Today will explain the connection of \( L(V,0) = 0 \) giving deformations on \( \varepsilon \otimes V \otimes 1 \) for \( V \) cuspidal auto. These are two parts to this:

1) \( L(V,0) = 0 \Rightarrow \text{existence of a (p-adically) holomorphic modular form} \)

   \text{with associated Galois rep. } \varepsilon \otimes V \otimes 1.

2) Deforming these to "special" modular forms.

Focus on 3 examples:

(a) \( \delta(-2m) = 0 \) if \( m > 1 \) and \( \delta(0) \neq 0 \).

(b) \( V = \chi \otimes V \), trivial char., \( \varepsilon \) newform

\[ L(V,0) = L(f,k), \quad \varepsilon(f,k) = -1. \]

(c) \( \varepsilon(f,k) = \pm 1 \).

For odd weight, there are no Selmer groups expected to be infinite.

(a) \( G_{2k}(\tau, s) = \sum_{\gamma \in \Gamma_0(2k)} \frac{|j(\gamma, \tau)|^{2k}}{|j(\gamma, \tau)|^5} \) \( \text{Re} \{k \} > 0 \).
$$\sum_{m=-\infty}^{\infty} C_m(s,y) e^{(mx)} (t = x + iy)$$

cm fact, one can compute

$$C_m(s,y) = \left( \frac{-1}{2\pi i} \right)^m \frac{\Gamma(1-5\frac{m}{2}) \Gamma \left( \frac{8s+4m}{2} \right) \Gamma \left( \frac{6-5-2m}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{8s}{2} \right) \Gamma \left( \frac{8s+2m}{2} \right) \Gamma \left( \frac{6-5-2m}{2} \right) \Gamma \left( \frac{1-5-2m}{2} \right)} y^{1-2k-5}$$

This is analytic at $s = 0 \land 2k > 2$.

Holomorphic in $t$ at $s = 0 \iff \delta(2-2k) = 0$. Thus, $G_{2k}(\tau, 0)$ is holomorphic in $t$ $\iff \delta(2-2k) = 0$ if $2k \geq 4$.

(One could actually use that $Hf(q, \varphi_1) = 0$ to show $\delta(0) = 0$!)

(b) $f$ at $2k$, trivial sign, $\delta(f, k) = -1 \Rightarrow \ell(V, \delta) = 0$.

cm this case, $F$ a Siegel modular form on $F$, $M$ at $k-1$.

That is (note: Chris used at $k$, but $s-k+1$ gives $2k \to k-1$)

- cuspidal
- holomorphic
- $L(s, \rho_1, F, \delta) = L(f, s) \delta(s-k+1) \delta(s-k+2)$ CAP form

$\rho \leftrightarrow F$

$\rho \leftrightarrow$ hol. cusp. rep.

$\circ \rho \leftrightarrow$ "minimally ramified" (Can shift all this to)

The factorization of the $L$-fun gives

$$\rho_{F} = \rho_{\delta} \otimes \varepsilon_{1-k} \otimes \varepsilon_{2-k}$$

Thus, $\rho_{F}(k) \cong \{0 \oplus \{0 \oplus \{z \oplus \{z \}} \} \}$

(c) Act

$$T_n = \left( \begin{array}{c|c} 1 \quad 1 \\ \hline -1 \quad n \end{array} \right) \in \text{GL}_2$$
\[ G_n = \text{unitary group associated skew Hermitian pair } (K^{2n}, T_n). \]
\[ R = K^{\text{alg}}. \]
\[ G_n (R) = \{ g \in GL_{2n} (K^{\text{alg}}) : g T_n b g^* = T_n \} \]
\[ G_n (R) \cong U(n,n) \]
\[ P_n = \text{stab. } g \cong 0 \otimes \cdots \otimes 0 \otimes K \quad P_n = \text{max } E - \text{ parabolici } \]
\[ M_n = \left\{ \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) : (A B) \in G_{n-1}, \quad t \in \text{Re} K^{\text{alg}} G_m \right\} \]
\[ \cong G_{n-1} \times \text{Re} K^{\text{alg}} G_m \]

\( N_n = \text{unipotent radical.} \)

\[ \text{Eisenstein series:} \]

\[ \pi \text{ cusp. rep. of } G_n (\mathbb{A}) , \text{ x idele class char. } \psi, \quad \psi^* \cong (\text{Re} K^{\text{alg}} G_m) (\mathbb{A}) \]
\[ \rho = \rho_{\pi,\psi} = \text{rep. of } P_n (\mathbb{A}) \text{ on } \mathcal{V}_{\pi} \]
\[ \rho (m g, t) \psi = \psi(c g) \times (\xi t) \psi \]
\[ s = \text{modulus character for } P_n , \quad s_n (m g, t) = 1 t^{n-1} \]

\[ I(\rho) \cong \psi \]
\[ E(\psi, g, s) = \sum_{Y \in P_n (\mathbb{A}) \setminus G_n (\mathbb{A})} \psi (Y g) \delta (Y g) s + 1/2 \]
\[ g \in G_n (\mathbb{A}), \quad s \in \mathbb{C} \]

(Note \( \psi (Y g) \) really takes values in \( \mathbb{C} \), so we always evaluate at 1 when defining a function.)

The interesting term is in the constant term. The only
interesting one is along $P_0$:

$$E_{P_0}(Q, g, s) = \Phi(Q)^{s+\frac{1}{2}} + M(Q)(Q)^{s-\frac{1}{2}}$$

$$\Phi = \otimes \Phi_v, \quad I(Q) = \otimes I(Q_v).$$

$$M(Q)(Q) = \otimes M_v(Q)(Q_v)$$

$$\text{for all places we have: this is } L\left(\frac{\nu}{2}, \frac{1}{2}, (2n-1), L\left(\frac{\nu}{2}, \frac{1}{2}, (2n-1) \right) \right).$$

The only way for $(*)$ to give a pole is if $g$ contains the Riemann zeta function.

Let $H = G_2, G = G_1$. If $\omega = 2^k$, trivial char, level $N$.

Associate to $f$ the form $\Phi(g) = j(g, \omega, \varphi)^{-2\omega} f(g, \omega, \varphi)$, $g = y_2 g_2$.

$GL_2(A) = GL_2(Q) \times GL_2(Q, \mathbb{R}) K_2(N)$.

$\Phi_{\text{idele class char. of } \mathbb{A}_K} \in \mathcal{S}_k. \quad \Phi_{\text{int.}}(2) = 2^{-k} \cdot \Phi_{\text{int.}}(2)$

$\Phi|_{\mathbb{A}^x} = 1.1^{-2\omega}$

$\mathcal{M}_K \to GL_2(\mathbb{A}_K)$

Write $g = x \cdot \Phi(y) = \Phi(x) \Phi(x) \to \mathcal{F}$ on $G(\mathbb{A})$.

$x = \Phi^{-1} \Phi^{-1} \cdot \Phi^{-1} \Phi^{-1}$

This $\Phi^{-1}$ is used to define the Eisenstein series.

We want to get $L(\nu, s) \delta(s-1) L(s, \chi)$ up to factors.
This H-T will predict the wt

\[ \text{HT-wt} \left( -\frac{1}{2}, \frac{1}{2} \right) -1 \ 0 \ \ {\text{as we want}}. \]

\[ \text{This tells where to look for the Eisenstein series.} \]

Back to the intertwining operators: in the case we get:

\[ 37 = \frac{1}{2} + k \]

\[ L \left( f, \frac{\varphi}{L}, \frac{1}{2} \right) \text{ s.c.} \text{ at } s = \frac{1}{2} + i \gamma. \]

This only works for \( 2k \leq 4 \), so excludes elliptic curves.

If sign of \( \varphi \) is \(-1\), can use \( p\)-adic family to get down to wt 2.

If sign is \(+1\), get \( \text{merely holo}\) \( \varphi \).