Math 852 Homework 13

1. Let $K_1$ and $K_2$ be purely inseparable extensions of a field $F$ with $K_1$ and $K_2$ contained in a large field. Show that $K_1K_2$ is purely inseparable over $F$.

2. Let $S_1$ and $S_2$ be two transcendence bases of $K/F$. Prove that $S_1$ and $S_2$ have the same cardinality. (Hint: Look back at how we showed two bases of a vector space had the same cardinality and use this replacement method to prove this result.)

3. Let $p$ be prime and let $K = \mathbb{F}_p(x, y)$ with $x$ and $y$ independent transcendentals over $\mathbb{F}_p$. Let $F = \mathbb{F}_p(x^p - x, y^p - x)$.

   (a) Prove that $[K : F] = p^2$ and the separable degree and inseparable degree of $K/F$ are both equal to $p$.

   (b) Prove that there is a subfield $E$ of $K$ containing $F$ which is purely inseparable over $F$ of degree $p$, so then $K$ is a separable extension of $E$ of degree $p$. (Hint: Let $s = x^p - x \in F$ and $t = y^p - x \in F$ and consider $s - t$.)

4. Let $F$ be a field of characteristic $p$ and $K$ a finite extension of $F$. Suppose that $[K : F]$ is relatively prime to $p$. Show that the extension $K/F$ is separable.

5. (a) Let $E = F(x)$ with $x$ transcendental over $F$. Let $K$ be a proper subfield of $E$ containing $F$. Show that $x$ is algebraic over $K$.

   (b) Let $E = F(x)$ with $x$ transcendental over $F$. Set $y = f(x)/g(x)$ with $f, g \in F[x]$ and $\gcd(f, g) = 1$. Let $n = \max(\deg f, \deg g)$ and suppose $n \geq 1$. Prove that $[F(x) : F(y)] = n$.

6. Let $k$ be a field with 4 elements, $t$ a transcendental over $k$, $F = k(t^4 + t)$, and $K = k(t)$.

   (a) Show that $[K : F] = 4$.

   (b) Show that $K$ is separable over $F$.

   (c) Show that $K$ is Galois over $F$. 
(d) Describe the lattice of subgroups of the Galois group and the corresponding lattice of subfields of $K$, giving each subfield in the form $k(r)$ for some rational function $r$.

7. (a) Find the splitting field of $f(x) = x^4 - 2$ over $\mathbb{Q}$. Call the splitting field $K$.

(b) Prove that $[K : \mathbb{Q}] = 8$. You may use this to conclude that $\text{Gal}(K/\mathbb{Q})$ has order 8 as well. (We may not have stated the Fundamental Theorem of Galois Theory yet, so just use it here.)

(c) Prove that there exists $\sigma \in \text{Gal}(K/\mathbb{Q})$ so that $\sigma(\sqrt[4]{2}) = i\sqrt[4]{2}$ and $\sigma(i) = i$. What is the order of $\sigma$?

(d) Let $\tau$ be restriction of complex conjugation to $K$. Show that $\tau \in \text{Gal}(K/\mathbb{Q})$. Show that

$$\text{Gal}(K/\mathbb{Q}) = \{\text{id}, \sigma, \sigma^2, \sigma^3, \tau, \tau\sigma, \sigma^2\tau, \sigma^3\tau\}.$$ 

What familiar group is this?

(e) Determine the fixed field of $\langle \sigma^2 \tau \rangle$.

(f) Let $E = \mathbb{Q}(\sqrt{2}, i)$. What is $\text{Gal}(K/E)$?

(g) Draw the subgroup and subfield diagrams for $K/\mathbb{Q}$ and $\text{Gal}(K/\mathbb{Q})$ as was done in class. You don’t need to justify all the containments, but label the degrees!